

## THE GROTHENDIECK INEQUALITY REVISITED

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ABSTRACT. The classical Grothendieck inequality is viewed as a statement about representations of functions of two variables over discrete domains by integrals of two-fold products of functions of one variable. An analogous statement is proved, concerning continuous functions of two variables over general topological domains. The main result is the construction of a norm-continuous map  $\Phi$  from  $l^2(A)$  into  $L^2(\Omega_A, \mathbb{P}_A)$ , where  $A$  is a set,  $\Omega_A = \{-1, 1\}^A$ , and  $\mathbb{P}_A$  is the uniform probability measure on  $\Omega_A$ , such that

$$\sum_{\alpha \in A} \mathbf{x}(\alpha) \overline{\mathbf{y}(\alpha)} = \int_{\Omega_A} \Phi(\mathbf{x}) \overline{\Phi(\mathbf{y})} d\mathbb{P}_A, \quad \mathbf{x} \in l^2(A), \mathbf{y} \in l^2(A), \quad (1)$$

and

$$\|\Phi(\mathbf{x})\|_{L^\infty} \leq K \|\mathbf{x}\|_2, \quad \mathbf{x} \in l^2(A), \quad (2)$$

for an absolute constant  $K > 1$ . ( $\Phi$  is non-linear, and does not commute with complex conjugation.) The Parseval-like formula in (1) is obtained by iterating the usual Parseval formula in a framework of harmonic analysis on dyadic groups. A modified construction implies a similar integral representation of the dual action between  $l^p$  and  $l^q$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ .

Variants of the Grothendieck inequality in higher dimensions are derived. These variants involve representations of functions of  $n$  variables in terms of functions of  $k$  variables,  $0 < k < n$ . Multilinear Parseval-like formulas are obtained, extending the bilinear formula in (1). The resulting formulas imply multilinear extensions of the Grothendieck inequality, and are used to characterize the feasibility of integral representations of multilinear functionals on a Hilbert space.

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## 1. Introduction

1.1. **The inequality.** We start with the infinite-dimensional Euclidean space

$$l^2 = l^2(\mathbb{N}) := \{\mathbf{x} = (\mathbf{x}(j))_{j \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sum_{j \in \mathbb{N}} |\mathbf{x}(j)|^2 < \infty\}, \quad (1.1)$$

equipped with the dot product

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{j \in \mathbb{N}} \mathbf{x}(j) \overline{\mathbf{y}(j)}, \quad \mathbf{x} \in l^2, \mathbf{y} \in l^2,$$

and Euclidean norm

$$\|\mathbf{x}\|_2 := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \left( \sum_{j \in \mathbb{N}} |\mathbf{x}(j)|^2 \right)^{\frac{1}{2}}, \quad \mathbf{x} \in l^2.$$

We let  $B_{l^2}$  denote the closed unit ball,

$$B_{l^2} := \{\mathbf{x} \in l^2 : \|\mathbf{x}\|_2 \leq 1\}.$$

The Grothendieck inequality in this setting is the assertion that there exists  $1 < K < \infty$  such that for every finite scalar array  $(a_{jk})$ ,

$$\sup \left\{ \left| \sum_{j,k} a_{jk} \langle \mathbf{x}_j, \mathbf{y}_k \rangle \right| : (\mathbf{x}_j, \mathbf{y}_k) \in (B_{l^2})^2 \right\} \leq K \sup \left\{ \left| \sum_{j,k} a_{jk} s_j t_k \right| : (s_j, t_k) \in [-1, 1]^2 \right\}. \quad (1.2)$$

An assertion equivalent to (1.2), couched in a setting of topological tensor products, had first appeared in Alexandre Grothendieck's landmark *Resumé* [19], and remained largely unnoticed until it was deconstructed and reformulated in [28] – another classic – as the inequality above. Since its reformulation, which became known as *the Grothendieck inequality*, it has been duly recognized as a fundamental statement, with diverse appearances and applications in functional, harmonic, and stochastic analysis, and recently also in theoretical physics and theoretical computer science. (See [31], and also Remark 2.9.ii in this work.)

The numerical value of the "smallest"  $K$  in (1.2), denoted by  $\mathcal{K}_G$  and dubbed *the Grothendieck constant*, is an open problem that to this day continues to attract interest. (For the latest on  $\mathcal{K}_G$ , see [13].)

**1.2. An integral representation.** Consider the infinite product space

$$\Omega_{B_{l^2}} := \{-1, 1\}^{B_{l^2}}, \quad (1.3)$$

equipped with the product topology and the sigma field generated by it, and then consider the coordinate functions  $r_{\mathbf{x}} : \Omega_{B_{l^2}} \rightarrow \{-1, 1\}$ ,  $\mathbf{x} \in B_{l^2}$ , defined by

$$r_{\mathbf{x}}(\omega) = \omega(\mathbf{x}), \quad \omega \in \{-1, 1\}^{B_{l^2}}, \quad \mathbf{x} \in B_{l^2}. \quad (1.4)$$

We refer to  $R_{B_{l^2}} := \{r_{\mathbf{x}} : \mathbf{x} \in B_{l^2}\}$  as a *Rademacher system indexed by  $B_{l^2}$* , and to its members as *Rademacher characters*; see §4 of this work.

**Proposition 1.1.** *The Grothendieck inequality (1.2) is equivalent to the existence of*

$$\lambda \in M(\Omega_{B_{l^2}} \times \Omega_{B_{l^2}}) \quad (= \{\text{complex measures on } \Omega_{B_{l^2}} \times \Omega_{B_{l^2}}\})$$

*such that*

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{\Omega_{B_{l^2}} \times \Omega_{B_{l^2}}} r_{\mathbf{x}}(\omega_1) r_{\mathbf{y}}(\omega_2) \lambda(d\omega_1, d\omega_2), \quad (\mathbf{x}, \mathbf{y}) \in B_{l^2} \times B_{l^2}, \quad (1.5)$$

*and the Grothendieck constant  $\mathcal{K}_G$  is the infimum of  $\|\lambda\|_M$  over all representations of the dot product by (1.5). ( $\|\lambda\|_M$  = total variation norm of  $\lambda$ .)*

*Proof.* We first verify (1.5)  $\Rightarrow$  (1.2). Let  $a = (a_{jk})$  be any finite scalar array, and denote by  $\|a\|_{\mathcal{F}_2}$  the supremum on the right side of (1.2). Assuming (1.5), let  $(\mathbf{x}_j)$  and  $(\mathbf{y}_k)$  be

arbitrary sequences of vectors in  $B_{l^2}$ , and then estimate

$$\begin{aligned}
\left| \sum_{j,k} a_{jk} \langle \mathbf{x}_j, \mathbf{y}_k \rangle \right| &= \left| \sum_{j,k} a_{jk} \int_{\Omega_{B_{l^2}} \times \Omega_{B_{l^2}}} r_{\mathbf{x}_j}(\omega_1) r_{\mathbf{y}_k}(\omega_2) \lambda(d\omega_1, d\omega_2) \right| \\
&\leq \int_{\Omega_{B_{l^2}} \times \Omega_{B_{l^2}}} \left| \sum_{j,k} a_{jk} r_{\mathbf{x}_j}(\omega_1) r_{\mathbf{y}_k}(\omega_2) \right| |\lambda|(d\omega_1, d\omega_2) \\
&\leq \|a\|_{\mathcal{F}_2} \|\lambda\|_M,
\end{aligned} \tag{1.6}$$

where  $\|\lambda\|_M$  denotes the total variation norm. We thus obtain (1.2) with  $\mathcal{K}_G \leq \|\lambda\|_M$ .

To verify (1.2)  $\Rightarrow$  (1.5), we first associate with every finite scalar array  $a = (a_{jk})$ , and sequences of vectors  $(\mathbf{x}_j)$  and  $(\mathbf{y}_k)$  in  $B_{l^2}$ , the *Walsh polynomial*

$$\hat{a} = \sum_{j,k} a_{jk} r_{\mathbf{x}_j} \otimes r_{\mathbf{y}_k}. \tag{1.7}$$

Note that  $\hat{a}$  is a continuous function on  $\Omega_{B_{l^2}} \times \Omega_{B_{l^2}}$ , and

$$\|\hat{a}\|_\infty = \|a\|_{\mathcal{F}_2}, \tag{1.8}$$

where  $\|\hat{a}\|_\infty$  denotes the supremum of  $\hat{a}$  over  $\Omega_{B_{l^2}} \times \Omega_{B_{l^2}}$ . The space of all such polynomials is norm-dense in  $C_{R_{B_{l^2}} \times R_{B_{l^2}}}(\Omega_{B_{l^2}} \times \Omega_{B_{l^2}})$ , the space of continuous functions on  $\Omega_{B_{l^2}} \times \Omega_{B_{l^2}}$  with spectrum in  $R_{B_{l^2}} \times R_{B_{l^2}}$ . Then, (1.2) is the statement that

$$\sum_{j,k} a_{jk} r_{\mathbf{x}_j} \otimes r_{\mathbf{y}_k} \mapsto \sum_{j,k} a_{jk} \langle \mathbf{x}_j, \mathbf{y}_k \rangle \tag{1.9}$$

determines a bounded linear functional on  $C_{R_{B_{l^2}} \times R_{B_{l^2}}}(\Omega_{B_{l^2}} \times \Omega_{B_{l^2}})$ , with norm bounded by  $\mathcal{K}_G$ . Therefore, by the Riesz Representation theorem and by the Hahn-Banach theorem, there exists  $\lambda \in M(\Omega_{B_{l^2}} \times \Omega_{B_{l^2}})$  such that (1.5) holds, and  $\|\lambda\|_M \leq \mathcal{K}_G$ .  $\square$

The implication (1.5)  $\Rightarrow$  (1.2) provides a template for direct proofs of the Grothendieck inequality: first establish an integral representation of the dot product, like the one in (1.5), and then verify the inequality by an "averaging" argument, as in (1.6); e.g., [28], [33], [4], [27]. Whereas there are other equivalent formulations of the inequality (e.g., see [31, §1, §2]), a representation of the dot product by an integral with uniformly bounded integrands is, arguably, the "closest" to it.

Building on ideas in [4], [5], and [6], we establish here integral representations that go a little further than (1.5), and imply a little more than (1.2).

**1.3. Parseval-like formulas.** Without a requirement that integrands be uniformly bounded, integral representations of the  $l^2$ -dot product are ubiquitous and indeed easy to produce: let  $\{f_k : k \in \mathbb{N}\}$  be an orthonormal system in  $L^2(\Omega, \mu)$ , where  $(\Omega, \mu)$  is a probability space, and let  $U$  be the map from  $l^2$  into  $L^2(\Omega, \mu)$  given by

$$U(\mathbf{x}) = \sum_k \mathbf{x}(k) f_k, \quad \mathbf{x} \in l^2, \quad (1.10)$$

whence (Parseval's formula),

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{\Omega} U(\mathbf{x}) \overline{U(\mathbf{y})} d\mu, \quad \mathbf{x} \in l^2, \quad \mathbf{y} \in l^2, \quad (1.11)$$

and

$$\|U(\mathbf{x})\|_{L^2} = \|\mathbf{x}\|_2, \quad \mathbf{x} \in l^2. \quad (1.12)$$

The map  $U$  is linear and (therefore) continuous.

The Grothendieck inequality – via Proposition 1.1 – is ostensibly the more stringent assertion, that there exist a probability space  $(\Omega, \mu)$  and a map  $\Phi$  from  $l^2$  into  $L^\infty(\Omega, \mu)$ , such that

$$\sum_k \mathbf{x}(k) \mathbf{y}(k) = \int_{\Omega} \Phi(\mathbf{x}) \Phi(\mathbf{y}) d\mu, \quad \mathbf{x} \in l^2, \quad \mathbf{y} \in l^2, \quad (1.13)$$

and

$$\|\Phi(\mathbf{x})\|_{L^\infty} \leq K \|\mathbf{x}\|_2, \quad \mathbf{x} \in l^2, \quad (1.14)$$

for some  $K > 1$ . Specifically, assuming that (1.2) holds, we take a complex measure  $\lambda \in M(\Omega_{B_{l^2}} \times \Omega_{B_{l^2}})$  supplied by Proposition 1.1, and let  $\psi$  be the Radon-Nikodym derivative of  $\lambda$  with respect to its total variation measure  $|\lambda|$ . For  $\mathbf{x} \in l^2$ , define

$$\sigma \mathbf{x} := \begin{cases} \mathbf{x} / \|\mathbf{x}\|_2 & \text{if } \mathbf{x} \neq \mathbf{0} \\ \mathbf{0} & \text{if } \mathbf{x} = \mathbf{0}, \end{cases} \quad (1.15)$$

and let  $\pi_i$  ( $i = 1, 2$ ) denote the canonical projections from  $\Omega_{B_{l^2}} \times \Omega_{B_{l^2}}$  onto  $\Omega_{B_{l^2}}$ ,

$$\pi_i(\omega_1, \omega_2) = \omega_i, \quad (\omega_1, \omega_2) \in \Omega_{B_{l^2}} \times \Omega_{B_{l^2}}.$$

Define

$$\phi_i : l^2 \rightarrow L^\infty(\Omega_{B_{l^2}} \times \Omega_{B_{l^2}}, |\lambda| / \|\lambda\|_M)$$

by

$$\phi_i(\mathbf{x}) = (\|\lambda\|_M \psi)^{\frac{1}{2}} \|\mathbf{x}\|_2 r_{\sigma \mathbf{x}} \circ \pi_i, \quad \mathbf{x} \in l^2, \quad i = 1, 2, \quad (1.16)$$

where  $r_{\sigma\mathbf{x}}$  are the coordinate functions given by (1.4), whence

$$\sum_k \mathbf{x}(k)\mathbf{y}(k) = \int_{\Omega} \phi_1(\mathbf{x})\phi_2(\mathbf{y})d\mu, \quad \mathbf{x} \in l^2, \mathbf{y} \in l^2, \quad (1.17)$$

with  $\Omega = \Omega_{B_{l^2}} \times \Omega_{B_{l^2}}$  and  $\mu = |\lambda|/\|\lambda\|_M$ . Then, applying polarization (with a bit of measure theory), we obtain the desired  $\Phi$  in (1.13) in terms of  $\phi_1$  and  $\phi_2$ . Notably, this  $\Phi$  is neither linear nor continuous.

The Grothendieck inequality *proper* guarantees the Parseval-like formula in (1.13) with a probability measure  $\mu$  on  $\Omega = \Omega_{B_{l^2}} \times \Omega_{B_{l^2}}$ , and a map

$$\Phi : l^2 \rightarrow L^\infty(\Omega_{B_{l^2}} \times \Omega_{B_{l^2}}, \mu)$$

that does not "detect" any of the ambient structures in  $l^2$  and  $L^\infty(\Omega, \mu)$ . A question arises: can we do better? That is, can a Parseval-like formula be derived with a more wieldy and canonical  $(\Omega, \mu)$ , and with  $\Phi$  that somehow "detects" the structures of its domain and range?

In the first part (§2 - §8), we represent general scalar-valued functions of two variables,  $x \in X$  and  $y \in Y$ , by integrals whose integrands are two-fold products of functions of one variable,  $x \in X$  and  $y \in Y$  separately. If  $X$  and  $Y$  are merely sets, then the Grothendieck inequality – seen as an integral representation of an inner product – plays a natural role. If  $X$  and  $Y$  are topological spaces, then to play that same role, the Grothendieck inequality needs an upgrade. The main result (Theorem 3.5) is the integral representation of the dot product

$$\sum_{\alpha \in A} \mathbf{x}(\alpha)\mathbf{y}(\alpha) = \int_{\Omega_A} \Phi(\mathbf{x})\Phi(\mathbf{y})d\mathbb{P}_A, \quad \mathbf{x} \in l^2(A), \mathbf{y} \in l^2(A), \quad (1.18)$$

where  $A$  is an infinite set,

$$\Omega_A := \{-1, 1\}^A,$$

$$\mathbb{P}_A := \text{uniform product measure (normalized Haar measure),}$$

and the map

$$\Phi : l^2(A) \rightarrow L^\infty(\Omega_A, \mathbb{P}_A) \quad (1.19)$$

is uniformly bounded on the unit ball of  $l^2(A)$  and also  $(l^2(A) \rightarrow L^2(\Omega_A, \mathbb{P}_A))$  - continuous.

We carry out the proof of Theorem 3.5 in a setting of harmonic analysis on dyadic groups. The construction of  $\Phi$  implicitly uses  $\Lambda(2)$ -uniformizability [4], a property of sparse spectral sets that manifests here through the use of Riesz products. Facts and tools drawn from harmonic analysis are reviewed as we move along (e.g., §4).

In §8, a modification of the proof of Theorem 3.5 yields a Parseval-like formula for  $\langle \mathbf{x}, \mathbf{y} \rangle$ ,  $\mathbf{x} \in l^p$ ,  $\mathbf{y} \in l^q$ ,  $1 \leq p \leq 2 \leq q \leq \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ . (Theorem 8.1).

**1.4. Multilinear Parseval-like formulas.** In the second part (§9 - §11), we represent functions of  $n$  variables ( $n \geq 2$ ) by integrals whose integrands involve functions of  $k$  variables,  $k < n$ , and in this context consider extensions of the (two-dimensional) Grothendieck inequality to dimensions greater than two.

First, following a "dual" view of (1.2) as the Parseval-like formula in (1.13), we derive analogous formulas in arbitrary dimensions. The general result (Theorem 11.3) is cast in a framework of *fractional Cartesian products*. It is proved by induction, with key steps provided by Theorem 3.5.

To illustrate the multilinear formulas of Theorem 11.3, we describe two archetypal instances. In the first, we take a simple extension of the (bilinear) dot product in  $l^2(A)$ ,

$$\Delta_n(\mathbf{x}_1, \dots, \mathbf{x}_n) = \sum_{\alpha \in A} \mathbf{x}_1(\alpha) \cdots \mathbf{x}_n(\alpha), \quad \mathbf{x}_i \in l^2(A), \quad i = 1, \dots, n. \quad (1.20)$$

With no additional requirements, integral representations of  $\Delta_n$ , which extend the usual Parseval formula, arise typically as follows. Consider the bounded linear map

$$U : l^2(A) \rightarrow L^2(\Omega_A, \mathbb{P}_A), \quad (1.21)$$

defined by

$$U(\mathbf{x}) = \sum_{\alpha \in A} \mathbf{x}(\alpha) r_\alpha, \quad \mathbf{x} \in l^2(A), \quad (1.22)$$

where the  $r_\alpha$  are Rademacher characters,

$$r_\alpha(\omega) = \omega(\alpha), \quad \omega \in \Omega_A := \{-1, 1\}^A, \quad \alpha \in A. \quad (1.23)$$

Then,

$$\begin{aligned} \Delta_n(\mathbf{x}_1, \dots, \mathbf{x}_n) &= \int_{(\omega_1, \dots, \omega_{n-1}) \in \Omega_A^{n-1}} \left( \prod_{i=1}^{n-1} U(\mathbf{x}_i)(\omega_i) \right) U(\mathbf{x}_n)(\omega_1 \cdots \omega_{n-1}) \, d\omega_1 \cdots d\omega_{n-1} \\ &= (U(\mathbf{x}_1) * \cdots * U(\mathbf{x}_n))(\mathbf{e}), \quad \mathbf{x}_1 \in l^2(A), \dots, \mathbf{x}_n \in l^2(A), \end{aligned} \quad (1.24)$$

where the integral above is performed with respect to the  $(n-1)$ -fold product of the Haar measure  $\mathbb{P}_A$  of the compact group  $\Omega_A$ , and is simply the  $n$ -fold convolution of  $U(\mathbf{x}_1), \dots, U(\mathbf{x}_n)$ , evaluated at the identity element  $\mathbf{e} \in \Omega_A$ ,

$$\mathbf{e}(\alpha) = 1, \quad \alpha \in A.$$



Adding the requirement that integrands be uniformly bounded, we obtain a nearly identical formula (Lemma 11.1),

$$\begin{aligned} \Delta_n(\mathbf{x}_1, \dots, \mathbf{x}_n) &= \int_{(\omega_1, \dots, \omega_{n-1}) \in \Omega_A^{n-1}} \left( \prod_{i=1}^{n-1} \Phi_n(\mathbf{x}_i)(\omega_i) \right) \Phi_n(\mathbf{x}_n)(\omega_1 \cdots \omega_{n-1}) d\omega_1 \cdots d\omega_{n-1} \\ &= (\Phi_n(\mathbf{x}_1) * \cdots * \Phi_n(\mathbf{x}_n))(\mathbf{e}), \quad \mathbf{x}_1 \in l^2(A), \dots, \mathbf{x}_n \in l^2(A), \end{aligned} \quad (1.25)$$

with the  $(l^2 \rightarrow L^2)$ -continuous map

$$\Phi_n : l^2(A) \rightarrow L^\infty(\Omega_A, \mathbb{P}_A), \quad (1.26)$$

which is produced by modifying the construction of  $\Phi$  in the proof of Theorem 3.5.

The second instance is the trilinear functional  $\eta$  on  $l^2(A^2) \times l^2(A^2) \times l^2(A^2)$ ,

$$\eta(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{(\alpha_1, \alpha_2, \alpha_3) \in A^3} \mathbf{x}(\alpha_1, \alpha_2) \mathbf{y}(\alpha_2, \alpha_3) \mathbf{z}(\alpha_1, \alpha_3), \quad (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in l^2(A^2) \times l^2(A^2) \times l^2(A^2). \quad (1.27)$$

To obtain a generic integral representation of  $\eta$ , with no restrictions on integrands, we take the bounded linear map  $U_2$  from  $l^2(A^2)$  into  $L^2(\Omega_A^2, \mathbb{P}_A^2)$  given by

$$U_2(\mathbf{x}) = \sum_{(\alpha_1, \alpha_2) \in A^2} \mathbf{x}(\alpha_1, \alpha_2) r_{\alpha_1} \otimes r_{\alpha_2}, \quad \mathbf{x} \in l^2(A^2), \quad (1.28)$$

and then observe

$$\begin{aligned} \eta(\mathbf{x}, \mathbf{y}, \mathbf{z}) &= \int_{(\omega_1, \omega_2, \omega_3) \in \Omega_A^3} U_2(\mathbf{x})(\omega_1, \omega_2) U_2(\mathbf{y})(\omega_2, \omega_3) U_2(\mathbf{z})(\omega_1, \omega_3) d\omega_1 d\omega_2 d\omega_3, \\ &\quad \mathbf{x} \in l^2(A^2), \mathbf{y} \in l^2(A^2), \mathbf{z} \in l^2(A^2). \end{aligned} \quad (1.29)$$

Adding the requirement that integrands be uniformly bounded, we have the map

$$\Phi_{(2,2)} : l^2(A^2) \rightarrow L^\infty(\Omega_A^2, \mathbb{P}_A^2), \quad (1.30)$$

which is  $(l^2(A^2) \rightarrow L^2(\Omega_A^2, \mathbb{P}_A^2))$  - continuous, and satisfies

$$\begin{aligned} \eta(\mathbf{x}, \mathbf{y}, \mathbf{z}) &= \int_{(\omega_1, \omega_2, \omega_3) \in \Omega_A^3} \Phi_{(2,2)}(\mathbf{x})(\omega_1, \omega_2) \Phi_{(2,2)}(\mathbf{y})(\omega_2, \omega_3) \Phi_{(2,2)}(\mathbf{z})(\omega_1, \omega_3) d\omega_1 d\omega_2 d\omega_3, \\ \mathbf{x} &\in l^2(A^2), \mathbf{y} \in l^2(A^2), \mathbf{z} \in l^2(A^2). \end{aligned} \tag{1.31}$$

The map  $\Phi_{(2,2)}$  is obtained by a two-fold iteration of Theorem 3.5. (See Remark 11.5.)

An  $n$ -linear version ( $n \geq 3$ ) of the trilinear  $\eta$  in (1.27) is given by

$$\begin{aligned} \eta_n(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) &= \sum_{(\alpha_1, \alpha_2, \dots, \alpha_n) \in A^n} \mathbf{x}_1(\alpha_1, \alpha_2) \mathbf{x}_2(\alpha_2, \alpha_3) \cdots \mathbf{x}_n(\alpha_n, \alpha_1), \\ (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) &\in l^2(A^2) \times l^2(A)^2 \times \cdots \times l^2(A^2), \end{aligned} \tag{1.32}$$

whose subsequent integral representation is given by

$$\begin{aligned} \eta_n(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) &= \int_{(\omega_1, \omega_2, \dots, \omega_n) \in \Omega_A^n} \Phi_{(2,2)}(\mathbf{x}_1)(\omega_1, \omega_2) \Phi_{(2,2)}(\mathbf{x}_2)(\omega_2, \omega_3) \cdots \Phi_{(2,2)}(\mathbf{x}_n)(\omega_n, \omega_1) d\omega_1 d\omega_2 \cdots d\omega_n, \\ (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) &\in l^2(A^2) \times l^2(A) \times \cdots \times l^2(A^2). \end{aligned} \tag{1.33}$$

The  $n$ -linear functional in (1.32) together with its integral representation in (1.33) play key roles in the Banach algebra of Hilbert-Schmidt operators. (See Remark 11.8.)

**1.5. Projective boundedness and projective continuity.** By the Grothendieck inequality (via Proposition 1.1), if  $\eta$  is a bounded bilinear functional on a Hilbert space  $H$ , then there exist bounded maps  $\phi_1$  and  $\phi_2$  from  $B_H$  (= closed unit ball in  $H$ ) into  $L^\infty(\Omega, \mu)$ , for some probability space  $(\Omega, \mu)$ , such that

$$\eta(\mathbf{x}, \mathbf{y}) = \int_{\Omega} \phi_1(\mathbf{x}) \phi_2(\mathbf{y}) d\mu, \quad (\mathbf{x}, \mathbf{y}) \in B_H \times B_H. \tag{1.34}$$

Theorem 3.5 asserts that we can design  $\phi_1$  and  $\phi_2$  to be also  $(H \rightarrow L^2)$ -continuous.

In the three-dimensional case, we have trilinear functionals on an infinite-dimensional Hilbert space that admit integral representations with uniformly bounded and continuous integrands (e.g.,  $\Delta_3$  in (1.24) and  $\eta$  in (1.27)). But there exist also bounded trilinear functionals that do not; "dually" stated, there exist bounded trilinear functionals  $\eta$ , produced first in [42], which fail a trilinear version of (1.2) with  $\eta$  in the role of the dot product. (See Remark 11.7.)

We are led to a definition: an  $n$ -linear functional  $\eta$  on a Hilbert space  $H$  is *projectively bounded* or (the stronger property) *projectively continuous*, if there exist a probability space  $(\Omega, \mu)$  and bounded maps or, respectively (the more stringent requirement), bounded and  $(H \rightarrow L^2(\Omega, \mu))$ -continuous maps,

$$\phi_i : B_H \rightarrow L^\infty(\Omega, \mu), \quad i = 1, \dots, n,$$

such that

$$\eta(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) = \int_{\Omega} \phi_1(\mathbf{x}_1) \cdots \phi_n(\mathbf{x}_n) d\mu, \quad (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) \in (B_H)^n. \quad (1.35)$$

(See Definition 10.1.)

For  $n = 1$  and  $n = 2$ , all bounded linear functionals and all bounded bilinear functionals on a Hilbert space are projectively continuous (Proposition 10.2, Theorem 3.5).

For  $n > 2$ , how to identify projectively bounded or projectively continuous  $n$ -linear functionals is an open-ended question. We tackle it here by analyzing kernels of functionals. Let  $H$  be a Hilbert space, and  $A$  an orthonormal basis in it. Given a bounded  $n$ -linear functional  $\eta$  on  $H$ , we consider its *kernel relative to  $A$* ,

$$\theta_{A,\eta}(\alpha_1, \dots, \alpha_n) := \eta(\alpha_1, \dots, \alpha_n), \quad (\alpha_1, \dots, \alpha_n) \in A^n, \quad (1.36)$$

and ask:

**Question 1.2.** *What properties of  $\theta_{A,\eta}$  imply that  $\eta$  is projectively continuous, or projectively bounded?*

If  $\eta$  is projectively bounded, then its kernel relative to a basis  $A$  can be represented as an integral of  $n$ -fold products of uniformly bounded functions of one variable. It is not known whether this condition alone is also sufficient (Problem 12.6).

In this work, building on the multilinear Parseval-like formulas (Theorem 11.3), we show that if the support of  $\theta_{A,\eta}$  is a *fractional Cartesian product* of a certain type, and  $\theta_{A,\eta}$  has an integral representation with uniformly bounded integrands, then  $\eta$  is projectively continuous (Theorem 10.11). In this case, the respective spaces of projectively bounded and projectively continuous functionals are equal – distinguished, possibly, by different numerical constants (Problem 12.5). I do not know whether this is always true, that a multilinear functional is projectively bounded if and only if it is projectively continuous (Problem 12.4).

We list some questions in §12. Some are stated in prior sections, and others will almost surely occur to the reader in the course of the narrative. The subject is open, and also open-ended, with loose ends and different directions to follow.

The prerequisites for the work are relatively minimal: basic knowledge of classical functional analysis and classical harmonic analysis should suffice. To the extent possible, I tried to be inclusive and self-contained (with apologies to experts).

**1.6. A personal note and acknowledgements.** My involvement with the Grothendieck inequality began in 1975, in connection with its roles in harmonic analysis. Since then, over the decades, I gradually meandered away from the subject. In 2005, I learnt of the interest in the Grothendieck inequality in the computer science community, and in particular of the results in [15]. My own interest in the inequality was rekindled, and reinforced by a subsequent result in [2], that the Grothendieck-like inequalities in [15] were in a certain sense optimal. The optimality, which at first blush appeared counter-intuitive to me, made crucial use of a phenomenon that had been first verified in [24]. (See Remark 2.9.ii.) I learnt more about these results from Noga Alon's lecture, and also from separate conversations with Assaf Naor and Boris Kashin, at the Lindenstrauss-Tzafriri retirement conference in June 2005. The integral representation formula in (3.13) was then derived (without the  $(l^2 \rightarrow L^2)$ -continuity of  $\Phi$ ), and noted specifically as a counterpoint to the result in [24]. Soon after my seminar at IAS in February 2006, where the aforementioned counterpoint was presented, I received an email from Noga Alon, that a similar formula could be deduced also from the proof of the Grothendieck inequality in [27].

Two and a half years later, following a seminar about integral representations of multilinear functionals (Paris, June 2008), Gilles Pisier pointed out to me the implication  $(1.2) \Rightarrow (1.5)$  (with a proof different from the argument given here), underlining that the integral representation of the dot product in (1.5) and the Grothendieck inequality are in fact equivalent. These two post-seminar remarks, by Alon and Pisier, sent me back to the drawing board, to study and further analyze integral representations of general functions, and indeed in large part determined the direction of this work.

## 2. Integral representations: the case of discrete domains

**2.1. First question.** Whether it is feasible to represent a given function in terms of simpler functions is a recurring theme that appears in various guises throughout mathematics. The precise formulation of the problem of course depends on the context. Here we consider the issue in a quasi-classical setting of functional analysis.

We begin with a question that – mainly during the 1960's and 1970's – motivated studies of tensor products in harmonic analysis; e.g., [39], [25, Ch. VIII], [18, Ch. 11].

**Question 2.1.** *Let  $f$  be a bounded scalar-valued function of two variables,  $x \in X$  and  $y \in Y$ . Are there families of scalar-valued functions of one variable,  $\{g_\omega\}_{\omega \in \Omega}$  and  $\{h_\omega\}_{\omega \in \Omega}$  defined*

respectively on  $X$  and  $Y$ , such that

$$\sum_{\omega \in \Omega} \|g_\omega\|_\infty \|h_\omega\|_\infty < \infty, \quad (2.1)$$

and

$$f(x, y) = \sum_{\omega \in \Omega} g_\omega(x) h_\omega(y), \quad (x, y) \in X \times Y? \quad (2.2)$$

( $\|\cdot\|_\infty$  denotes, here and throughout, the usual sup-norm over an underlying domain.)

We refer to the functions  $g_\omega$  and  $h_\omega$ , indexed by  $\omega \in \Omega$  and defined respectively on  $X$  and  $Y$ , as *representing functions* of  $f$ . In the next section  $X$  and  $Y$  will have additional structures, but at this point, until further notice, these domains are viewed merely as sets. The constraint in (2.1) implies that the sum on the right side of (2.2) converges absolutely, and also that  $\Omega$  may as well be at most countable. Later we will consider more general representations and constraints, wherein the sums in (2.1) and (2.2) are replaced by integrals, and indexing sets are measure spaces. If constraints on representing functions are removed altogether, then the answer to Question 2.1 will always be affirmative (and also uninteresting): for, in the absence of constraints, given a scalar-valued function  $f$  on  $X \times Y$ , we can take the indexing set  $\Omega$  to be  $Y$ , and then let

$$g_y(x) = f(x, y), \quad x \in X, \quad y \in Y,$$

and for  $y \in Y$  and  $y' \in Y$ ,

$$h_{y'}(y) = \begin{cases} 0 & \text{if } y' \neq y \\ 1 & \text{if } y' = y. \end{cases}$$

Nowadays it is practically folklore – arguably originating in Littlewood’s classic paper [29] – that if  $X$  and  $Y$  are infinite, then there is an abundance of functions that cannot be represented by (2.2) under the constraint in (2.1). To illustrate how such functions can arise, we take

$$f(x, k) = e^{ixk}, \quad (x, k) \in [0, 2\pi] \times \mathbb{Z}. \quad (2.3)$$

An affirmative answer here to Question 2.1 would mean that there exist families of representing functions  $\{g_n\}_{n \in \mathbb{N}}$  and  $\{h_n\}_{n \in \mathbb{N}}$  defined respectively on  $[0, 2\pi]$  and  $\mathbb{Z}$ , such that

$$e^{ixk} = \sum_{n \in \mathbb{N}} g_n(x) h_n(k), \quad (x, n) \in [0, 2\pi] \times \mathbb{Z}, \quad (2.4)$$

and

$$\sum_{n \in \mathbb{N}} \|g_n\|_\infty \|h_n\|_\infty = c < \infty. \quad (2.5)$$

With this assumption, if a finite scalar array  $(a_{jk})_{j,k}$  satisfies

$$\sup \left\{ \left| \sum_{j,k} a_{jk} s_j t_k \right| : |s_j| \leq 1, |t_k| \leq 1 \right\} \leq 1, \quad (2.6)$$

then for all  $x_j \in [0, 2\pi]$ ,  $j = 1, \dots$ ,

$$\left| \sum_{j,k} a_{jk} e^{ix_j k} \right| \leq c. \quad (2.7)$$

Now observe that for every positive integer  $N$ , the array (a rescaled Fourier matrix)

$$a_{jk} = N^{-\frac{3}{2}} e^{\frac{-2\pi i j k}{N}}, \quad (j, k) \in [N]^2, \quad (2.8)$$

satisfies (2.6). Taking  $N > c^2$  and  $x_j = 2\pi i j / N$  for  $j \in [N]$ , we have

$$\sum_{j,k=1}^N a_{jk} e^{ix_j k} = N^{\frac{1}{2}} > c,$$

which contradicts (2.7), thus proving that  $f$  in (2.3) cannot be represented by (2.4) under the constraint in (2.5).

The preceding argument, verifying a negative answer to Question 2.1, made implicit use of a duality between two norms. The first,  $\|f\|_{\mathcal{V}_2(X \times Y)}$  for a given scalar-valued function  $f$  on  $X \times Y$ , is the infimum of the left side of (2.1) taken over all representations in (2.2). (Here and throughout, in the absence of ambiguity, the underlying  $X$  and  $Y$  will be suppressed from the notation; e.g.,  $\|f\|_{\mathcal{V}_2}$  will stand for  $\|f\|_{\mathcal{V}_2(X \times Y)}$ .) With this norm and point-wise multiplication on  $X \times Y$ , the resulting space

$$\mathcal{V}_2(X \times Y) = \{f : \|f\|_{\mathcal{V}_2} < \infty\}$$

is a Banach algebra. ( $\mathcal{V}$  is for Varopoulos; e.g., see [39], [18, Ch. 11].) The second norm,  $\|a\|_{\mathcal{F}_2}$  for a given scalar array  $a = (a_{xy})_{x \in X, y \in Y}$ , is defined as the supremum of

$$\left| \sum_{(x,y) \in S \times T} a_{xy} g(x) h(y) \right|$$

taken over all finite rectangles  $S \times T \subset X \times Y$ ,  $g \in B_{l^\infty(X)}$  and  $h \in B_{l^\infty(Y)}$ . (Here and throughout,  $B_E$  denotes the closed unit ball in a normed linear space  $E$ , and  $l^\infty(D)$  denotes the space of bounded scalar-valued functions on a domain  $D$ .) Equipped with the  $\mathcal{F}_2$ -norm and point-wise multiplication on  $X \times Y$ ,

$$\mathcal{F}_2(X \times Y) := \{a = (a_{xy})_{x \in X, y \in Y} : \|a\|_{\mathcal{F}_2} < \infty\} \quad (2.9)$$

is a Banach algebra. ( $\mathcal{F}$  is for Fréchet; e.g., see [17], [7, Ch. I].) Within the broader context of topological tensor products, the  $\mathcal{V}_2$ -norm and the  $\mathcal{F}_2$ -norm are instances, respectively, of *projective* and *injective* cross-norms; e.g., see [36], [20].

The duality between the  $\mathcal{V}_2$ -norm and the  $\mathcal{F}_2$ -norm, expressed by Lemma 2.2 below, is at the foundation of a large subject. This duality had been noted and investigated first by Schatten in the 1940's, e.g., [35], and was later extended and expansively studied by Grothendieck during the 1950's, e.g., [19], [20]. Since the mid-1960's, the lemma has reappeared in various forms and settings, and has become folklore. I learnt it nearly four decades ago in a specific context of harmonic analysis, e.g., [3].

**Lemma 2.2.** *If  $f \in \mathcal{V}_2(X \times Y)$ , then*

$$\|f\|_{\mathcal{V}_2} = \sup \left\{ \left| \sum_{(x,y) \in S \times T} a_{x,y} f(x,y) \right| : a \in B_{\mathcal{F}_2}, \text{ finite rectangles } S \times T \subset X \times Y \right\}. \quad (2.10)$$

(For proof, see Remark 2.5.ii.)

**2.2. Second question.** The converse to Lemma 2.2 is false: if  $X$  and  $Y$  are infinite, then there exist  $f \in l^\infty(X \times Y)$  for which the right side of (2.10) is finite, but  $f \notin \mathcal{V}_2(X \times Y)$ . (How these  $f$  arise is briefly explained in Remark 2.5.iii below.) However, if we broaden Question 2.1 (but keep to its original intent), then we are led exactly to those functions  $f$  for which the right side of (2.10) is finite.

**Question 2.3.** *Given  $f \in l^\infty(X \times Y)$ , can we find a probability space  $(\Omega, \mu)$  and representing functions indexed by it,  $\{g_\omega\}_{\omega \in \Omega}$  and  $\{h_\omega\}_{\omega \in \Omega}$  defined respectively on  $X$  and  $Y$ , such that for every  $(x, y) \in X \times Y$ ,*

$$\omega \mapsto g_\omega(x), \quad \omega \mapsto h_\omega(y), \quad \omega \mapsto \|g_\omega\|_\infty, \quad \omega \mapsto \|h_\omega\|_\infty, \quad \omega \in \Omega, \quad (2.11)$$

*determine  $\mu$ -measurable functions on  $\Omega$ ,*

$$\int_{\Omega} \|g_\omega\|_\infty \|h_\omega\|_\infty \mu(d\omega) < \infty, \quad (2.12)$$

*and*

$$f(x, y) = \int_{\Omega} g_\omega(x) h_\omega(y) \mu(d\omega)? \quad (2.13)$$

*(We refer to the probability space  $(\Omega, \mu)$  as an indexing space, and to the probability measure  $\mu$  as an indexing measure.)*

There exist functions of two variables that cannot be represented by (2.13) under the constraint in (2.12); e.g., the function  $f$  in (2.3), with the same proof. Question 2.3 obviously subsumes Question 2.1: an affirmative to Question 2.1 implies an affirmative to Question 2.3. But the converse is false, as per Remark 2.5.iii below. To formalize matters, given

$f \in l^\infty(X \times Y)$ , we define  $\|f\|_{\tilde{\mathcal{V}}_2}$  to be the infimum of the left side of (2.12) over all representations of  $f$  by (2.13), and then let

$$\tilde{\mathcal{V}}_2(X \times Y) = \{f \in l^\infty(X \times Y) : \|f\|_{\tilde{\mathcal{V}}_2} < \infty\}.$$

One can verify directly that  $\|\cdot\|_{\tilde{\mathcal{V}}_2}$  is a norm, and that  $(\tilde{\mathcal{V}}_2(X \times Y), \|\cdot\|_{\tilde{\mathcal{V}}_2})$  is a Banach algebra with point-wise multiplication on  $X \times Y$ . (E.g., see Proposition 3.3 in the next section.) This follows also as a corollary to a characterization of  $\tilde{\mathcal{V}}_2(X \times Y)$  as the dual space of  $\mathcal{F}_2(X \times Y)$ . To state and prove this characterization, we use an alternate definition of  $\mathcal{F}_2(X \times Y)$  cast in a harmonic analysis framework.

Given a set  $A$ , we define the Rademacher system indexed by  $A$

$$R_A = \{r_\alpha\}_{\alpha \in A}$$

to be the set of coordinate functions on  $\{-1, 1\}^A := \Omega_A$ ,

$$r_\alpha(\omega) = \omega(\alpha), \quad \omega \in \{-1, 1\}^A, \quad \alpha \in A.$$

We view the product space  $\Omega_A$  as a compact Abelian group with the product topology and coordinate-wise multiplication, and view  $R_A$  as a set of independent characters on it. (E.g., see [7, p. 139].)

Next we consider the set of characters

$$R_X \times R_Y = \{r_x \otimes r_y\}_{(x,y) \in X \times Y}$$

on the compact Abelian group  $\Omega_X \times \Omega_Y$ , where  $r_x \otimes r_y$  is the  $\{-1, 1\}$ -valued function on  $\Omega_X \times \Omega_Y$ , defined by

$$r_x \otimes r_y(\omega_1, \omega_2) = r_x(\omega_1)r_y(\omega_2), \quad (\omega_1, \omega_2) \in \Omega_X \times \Omega_Y.$$

Given a scalar array  $a = (a_{xy})_{(x,y) \in X \times Y}$ , we identify it formally with the Walsh series

$$\hat{a} \sim \sum_{(x,y) \in X \times Y} a_{xy} r_x \otimes r_y$$

(a distribution on  $\Omega_X \times \Omega_Y$ ), and observe that

$$a \in \mathcal{F}_2(X \times Y) \iff \hat{a} \in C_{R_X \times R_Y}(\Omega_X \times \Omega_Y), \quad (2.14)$$

where  $C_{R_X \times R_Y}(\Omega_X \times \Omega_Y) = \{\text{continuous functions on } \Omega_X \times \Omega_Y \text{ with spectra in } R_X \times R_Y\}$ ; e.g., see [7, Ch. IV, Ch. VII], and also §4 in this work. Specifically,

$$\|\hat{a}\|_\infty \leq \|a\|_{\mathcal{F}_2} \leq 4\|\hat{a}\|_\infty. \quad (2.15)$$

Therefore, the dual space of  $\mathcal{F}_2(X \times Y)$  can be identified (via (2.14) and Parseval's formula) with the dual space of  $C_{R_X \times R_Y}(\Omega_X \times \Omega_Y)$ , which (via the Riesz representation theorem and the Hahn-Banach theorem) is the algebra of restrictions to  $R_X \times R_Y$  of transforms of regular complex measures on  $\Omega_X \times \Omega_Y$ . This restriction algebra is denoted

$$B(R_X \times R_Y) := \widehat{M}(\Omega_X \times \Omega_Y) / \{\hat{\lambda} \in \widehat{M}(\Omega_X \times \Omega_Y) : \hat{\lambda} = 0 \text{ on } R_X \times R_Y\}.$$



**Proposition 2.4.**

$$\tilde{\mathcal{V}}_2(X \times Y) = B(R_X \times R_Y).$$

*Proof.* If  $f \in B(R_X \times R_Y)$ , then (by definition) there exists a regular complex measure  $\lambda$  on  $\Omega_X \times \Omega_Y$  such that

$$f(x, y) = \int_{\Omega_X \times \Omega_Y} r_x(\omega_1) r_y(\omega_2) \lambda(d\omega_1, d\omega_2), \quad (x, y) \in X \times Y,$$

which implies  $f \in \tilde{\mathcal{V}}_2(X \times Y)$ : the indexing space is  $(\Omega_X \times \Omega_Y, \frac{|\lambda|}{\|\lambda\|})$ ;  $|\lambda|$  is the *total variation measure* of  $\lambda$ ;  $\|\lambda\| = |\lambda|(\Omega_X \times \Omega_Y)$ , and representing functions are given by

$$\begin{aligned} g_{\omega_1 \omega_2}(x) &= r_x(\omega_1) \\ h_{\omega_1 \omega_2}(y) &= \|\lambda\| \frac{d\lambda}{d|\lambda|}(\omega_1, \omega_2) r_y(\omega_2), \quad (\omega_1, \omega_2) \in \Omega_X \times \Omega_Y, \quad (x, y) \in X \times Y. \end{aligned}$$

Suppose  $f \in \tilde{\mathcal{V}}_2(X \times Y)$ . Then, we have an indexing space  $(\Omega, \mu)$  and families of representing functions  $\{g_\omega\}_{\omega \in \Omega}$  and  $\{h_\omega\}_{\omega \in \Omega}$  defined respectively on  $X$  and  $Y$ , such that (2.11), (2.12), and (2.13) hold. Let  $\hat{a}$  be a Walsh polynomial in  $C_{R_X \times R_Y}(\Omega_X \times \Omega_Y)$ ,

$$\hat{a} = \sum_{(x,y) \in X \times Y} a_{xy} r_x \otimes r_y,$$

where  $(a_{xy})_{(x,y) \in X \times Y}$  is a scalar array with finite support in  $X \times Y$ . Then, by (2.13), (2.12), and (2.15),

$$\begin{aligned} \left| \sum_{(x,y) \in X \times Y} a_{xy} f(x, y) \right| &= \left| \sum_{(x,y) \in X \times Y} a_{xy} \int_{\Omega} g_\omega(x) h_\omega(y) \mu(d\omega) \right| \\ &\leq \int_{\Omega} \left| \sum_{(x,y) \in X \times Y} a_{xy} g_\omega(x) h_\omega(y) \right| \mu(d\omega) \\ &\leq 4 \|\hat{a}\|_{\infty} \int_{\Omega} \|g_\omega\|_{\infty} \|h_\omega\|_{\infty} \mu(d\omega) \leq 4 \|\hat{a}\|_{\infty} \|f\|_{\mathcal{V}_2}. \end{aligned} \tag{2.16}$$

Because Walsh polynomials are norm-dense in  $C_{R_X \times R_Y}(\Omega_X \times \Omega_Y)$ , the estimate in (2.16) implies that  $f$  determines a bounded linear functional on  $C_{R_X \times R_Y}(\Omega_X \times \Omega_Y)$ , and hence  $f \in B(R_X \times R_Y)$ . □

**Remark 2.5.**

i. The simple proof of Proposition 2.4 is similar to the proof of Proposition 1.1. The

compact Abelian group  $\{-1, 1\}^A$  in the proof above can be replaced, with nearly identical effect, by the compact Abelian group

$$\{e^{it} : t \in [0, 2\pi)\}^A$$

(equipped with coordinate-wise multiplication, and the usual topology on each coordinate). The Rademacher system becomes the Steinhaus system (cf. [7, p. 29]), and the norm equivalence in (2.15) becomes the isometry

$$\|\hat{a}\|_\infty = \|a\|_{\mathcal{F}_2}.$$

A modified proof of Proposition 2.4 in this setting yields

$$\|f\|_{\tilde{\mathcal{V}}_2} = \sup \left\{ \left| \sum_{(x,y) \in S \times T} a_{xy} f(x, y) \right| : \|a\|_{\mathcal{F}_2} \leq 1, \text{ finite } S \times T \subset X \times Y \right\}.$$

**ii.** By an elementary argument (involving measure theory),

$$\|f\|_{\mathcal{V}_2(X \times Y)} = \|f\|_{\tilde{\mathcal{V}}_2(X \times Y)}, \quad f \in \mathcal{V}_2(X \times Y), \quad (2.17)$$

and hence Lemma 1.1. The equality of norms in (2.17) can be observed also by noting that  $\mathcal{V}_2(X \times Y)$  equals  $B_d(R_X \times R_Y)$ , the algebra of restrictions to  $R_X \times R_Y$  of transforms of *discrete* measures on  $\Omega_X \times \Omega_Y$ . (See [41].)

**iii.** In the language of harmonic analysis (via Proposition 2.4), the proper inclusion

$$\tilde{\mathcal{V}}_2(X \times Y) \subsetneq l^\infty(X \times Y) \quad (2.18)$$

for infinite  $X$  and  $Y$  is the assertion that  $R_X \times R_Y$  is not a Sidon set, a classical fact that can be verified in several ways; e.g., via the example in (2.3). Also in this language, the proper inclusion

$$\mathcal{V}_2(X \times Y) \subsetneq \tilde{\mathcal{V}}_2(X \times Y)$$

becomes the assertion

$$B_d(R_X \times R_Y) \subsetneq B(R_X \times R_Y), \quad (2.19)$$

which is proved in [41], via [40]. The proper inclusion in (2.19) can be deduced also from (2.18) and a characterization of Sidonicity found in [11, p. 118].

**iv.** By Proposition 2.4, if  $f \in \tilde{\mathcal{V}}_2(X \times Y)$ , then there exists  $\lambda \in M(\Omega_X \times \Omega_Y)$  such that

$$f(x, y) = \hat{\lambda}(r_x \otimes r_y), \quad (x, y) \in X \times Y.$$

This implies that in the definition of the  $\|\cdot\|_{\tilde{\mathcal{V}}_2}$ -norm, the constraint in (2.12) (absolute integrability) can be replaced by the stronger condition ( $L^\infty$ -boundedness)

$$\sup_{x \in X, y \in Y} \operatorname{ess\,sup}_{\omega \in \Omega} |g_\omega(x) h_\omega(y)| < \infty. \quad (2.20)$$

Specifically, the  $\|\cdot\|_{\tilde{\mathcal{V}}_2}$ -norm is equivalent to the norm obtained by taking the infimum of the left side of (2.20) over all representations in (2.13).

**2.3. Third question and Grothendieck's *théorème fondamental*.** Next we replace (2.12) in Question 2.3 with a weaker constraint:

**Question 2.6.** *Given  $f \in l^\infty(X \times Y)$ , can we find an indexing space  $(\Omega, \mu)$ , and representing functions indexed by it,  $\{g_\omega\}_{\omega \in \Omega}$  and  $\{h_\omega\}_{\omega \in \Omega}$  defined respectively on  $X$  and  $Y$ , so that for every  $(x, y) \in X \times Y$ ,*

$$\omega \mapsto g_\omega(x), \quad \omega \mapsto h_\omega(y), \quad \omega \in \Omega,$$

*determine elements in  $L^2(\Omega, \mu)$ ,*

$$\sup_{x \in X, y \in Y} \left( \int_{\Omega} |g_\omega(x)|^2 \mu(d\omega) \right)^{\frac{1}{2}} \left( \int_{\Omega} |h_\omega(y)|^2 \mu(d\omega) \right)^{\frac{1}{2}} < \infty, \quad (2.21)$$

*and*

$$f(x, y) = \int_{\Omega} g_\omega(x) h_\omega(y) \mu(d\omega)? \quad (2.22)$$

The constraint in (2.21) insures that the right side of (2.22) is well defined, and appears weaker (via Remark 2.5.iv above) than the condition in (2.12). The two conditions (2.12) and (2.21) are, respectively, the "maximal" and "minimal" constraints that guarantee convergence of the respective integral representations in (2.13) and (2.22). (See also Remark 2.9.i below.)

Grothendieck's celebrated theorem, dubbed *le théorème fondamental de la théorie métrique des produits tensoriels* in [19, p. 59], is in effect the statement that the aforementioned constraints are equivalent. That is, for every function on  $X \times Y$ , the respective answers to Questions 2.3 and 2.6 are identical: a function of two variables can be represented by (2.13) under the constraint in (2.12) if and only if it can be represented by (2.22) under the constraint in (2.21). To state this precisely, given  $f \in l^\infty(X \times Y)$ , we define  $\|f\|_{\mathcal{G}_2}$  to be the infimum of the left side of (2.21) over all representations of  $f$  by (2.22), and let

$$\mathcal{G}_2(X \times Y) = \{f \in l^\infty(X \times Y) : \|f\|_{\mathcal{G}_2} < \infty\}.$$

(We omit the verification that  $(\mathcal{G}_2(X \times Y), \|\cdot\|_{\mathcal{G}_2})$  is a Banach algebra; see the theorem below, and also Proposition 3.3 in the next section.)

**Theorem 2.7** (a version of Grothendieck's *théorème fondamental*).

$$\mathcal{G}_2(X \times Y) = \tilde{\mathcal{V}}_2(X, Y).$$

To verify  $\tilde{\mathcal{V}}_2(X, Y) \subset \mathcal{G}_2(X \times Y)$ , simply note that if  $f \in \tilde{\mathcal{V}}_2(X \times Y)$ , then (by Proposition 2.4) there exists  $\lambda \in M(\Omega_X \times \Omega_Y)$ , such that

$$f(x, y) = \hat{\lambda}(r_x \otimes r_y), \quad (x, y) \in X \times Y,$$

and (by Remark 2.5.iv) we have the norm estimate

$$\|f\|_{\mathcal{G}_2} \leq \|f\|_{\tilde{\mathcal{V}}_2}, \quad f \in \tilde{\mathcal{V}}_2(X, Y).$$

The reverse inclusion

$$\mathcal{G}_2(X \times Y) \subset \tilde{\mathcal{V}}_2(X, Y),$$

together with the norm estimate

$$\|f\|_{\mathcal{G}_2} \leq 1 \Rightarrow \|f\|_{\tilde{\mathcal{V}}_2} \leq K, \quad (2.23)$$

for a universal constant  $1 < K < \infty$ , is the essence of Grothendieck's theorem.

Focusing on (2.23), note that  $\|f\|_{\mathcal{G}_2} \leq 1$  means that there exist maps from  $X$  and  $Y$  into the unit ball  $B_H$  of a Hilbert space  $H$ ,

$$\begin{aligned} x &\mapsto \mathbf{u}_x, & x \in X, & \mathbf{u}_x \in B_H, \\ y &\mapsto \mathbf{v}_y, & y \in Y, & \mathbf{v}_y \in B_H, \end{aligned} \quad (2.24)$$

such that

$$f(x, y) = \langle \mathbf{u}_x, \mathbf{v}_y \rangle_H, \quad (x, y) \in X \times Y, \quad (2.25)$$

where  $\langle \cdot, \cdot \rangle_H$  is the inner product on  $H$ . The norm estimate in (2.23) then becomes (via Proposition 2.4) the Grothendieck inequality,

$$\left| \sum_{x,y} a_{xy} \langle \mathbf{u}_x, \mathbf{v}_y \rangle_H \right| \leq K \|a\|_{\mathcal{F}_2}$$

for all finitely supported scalar arrays  $(a_{xy})_{x \in X, y \in Y}$ ; cf. (1.2). Therefore, (2.23) is naturally equivalent to the statement (again via Proposition 2.4) that the inner product on  $H$ , viewed as a function on  $B_H \times B_H$ , is in  $\tilde{\mathcal{V}}_2(B_H, B_H)$ :

**Theorem 2.8** (yet another version of *le théorème fondamental*). *Let  $H$  be an infinite-dimensional Hilbert space with inner product  $\langle \cdot, \cdot \rangle_H$ . Then,*

$$\|\langle \cdot, \cdot \rangle_H\|_{\tilde{\mathcal{V}}_2(B_H, B_H)} := \mathcal{K}_G < \infty, \quad (2.26)$$

where  $\mathcal{K}_G$  is independent of  $H$ .

The constant in (2.26) is *the Grothendieck constant*. The determinations of its two distinct values  $\mathcal{K}_G^{\mathbb{C}}$  and  $\mathcal{K}_G^{\mathbb{R}}$ , which correspond to the scalar fields  $\mathbb{C}$  or  $\mathbb{R}$  in the definition of  $\tilde{\mathcal{V}}_2$ , are open problems. To date, the best published estimates of  $\mathcal{K}_G^{\mathbb{C}}$  and  $\mathcal{K}_G^{\mathbb{R}}$  are found in [27] and [21], respectively; a recently improved estimate of  $\mathcal{K}_G^{\mathbb{R}}$  appears in [13].

**Remark 2.9.**

i. The  $L^2 - L^2$  constraint in (2.21), which guarantees (via Cauchy-Schwarz) the existence of the integral in (2.22), can be replaced by an  $L^p - L^q$  constraint ( $\frac{1}{p} + \frac{1}{q} = 1$ ,  $2 \geq p \leq \infty$ ), which produces (via Hölder) the same effect. Then, modifying accordingly Question 2.6, we look for integral representations of  $f \in l^\infty(X \times Y)$  under the constraint

$$\sup_{x \in X, y \in Y} \|g_\omega(x)\|_{L^p(\mu)} \|h_\omega(y)\|_{L^q(\mu)} < \infty.$$

We consider the space  $\mathcal{G}_{(p,q)}(X \times Y)$ , whose definition is analogous to that of  $\mathcal{G}_2(X \times Y)$ , and ask whether a Grothendieck-type theorem holds here as well. That is, do we have

$$\mathcal{G}_{(p,q)}(X \times Y) = \mathcal{V}_2(X \times Y)?$$

(Cf. Theorem 2.7.)

For  $p > 2$ , and infinite  $X$  and  $Y$ , the answer is *no*. The proof, like that of

$$\mathcal{V}_2(X \times Y) \subsetneq l^\infty(X \times Y),$$

makes use of the finite Fourier matrix. For an integer  $N > 0$ , we take  $X = Y = [N]$ , and indexing space  $\Omega = [N]$  with the uniform probability measure  $\mu$  on it, i.e.,

$$\mu(\{\omega\}) = \frac{1}{N}, \quad \omega \in [N].$$

We take representing functions

$$g_\omega(j) = e^{\frac{-2\pi i j \omega}{N}}, \quad \omega \in \Omega, \quad j \in X,$$

$$h_\omega(k) = \begin{cases} 0 & \text{if } \omega \neq k \\ N^{\frac{1}{q}} & \text{if } \omega = k, \end{cases} \quad \omega \in \Omega, \quad k \in Y,$$

and let

$$f(j, k) := \int_{\Omega} g_\omega(j) h_\omega(k) \mu(d\omega) = N^{-\frac{1}{p}} e^{\frac{-2\pi i j k}{N}}, \quad (j, k) \in X \times Y.$$

Because

$$\sup_{j \in X, k \in Y} \left( \int_{\Omega} |g_\omega(j)|^p \mu(d\omega) \right)^{\frac{1}{p}} \left( \int_{\Omega} |h_\omega(k)|^q \mu(d\omega) \right)^{\frac{1}{q}} = 1,$$

we have  $\|f\|_{\mathcal{G}_{(p,q)}} \leq 1$ . By duality, via Proposition 2.4 and the rescaled Fourier matrix in (2.8), we also have

$$\|f\|_{\hat{\mathcal{V}}_2} \geq \frac{1}{N^{\frac{3}{2}}} \sum_{j,k=1}^N f(j, k) e^{\frac{2\pi i j k}{N}} = N^{\frac{1}{2} - \frac{1}{p}} \xrightarrow{N \rightarrow \infty} \infty,$$

and conclude that if  $X$  and  $Y$  are infinite sets, then

$$\mathcal{V}_2(X \times Y) \subsetneq \mathcal{G}_{(p,q)}(X \times Y).$$

ii. Grothendieck's *théorème fondamental* had been stated as a "factorization" theorem in a framework of topological tensor products [19], and was later reformulated as an elementary inequality in a context of Banach space theory [28]. Since its reformulation, which made *le théorème fondamental* accessible to a larger public, Grothendieck's theorem has evolved, migrating further and farther into various diverse settings. (See [31].)

Recently, the Grothendieck inequality and variants of it have appeared in studies of algorithmic complexity in a context of theoretical computer science; e.g., [2], [15], [1], [37], [26]. These studies began with the following inequality [15]. Let  $(a_{jk})_{(j,k) \in \mathbb{N}^2}$  be an infinite matrix with real-valued entries, and  $a_{jj} = 0$  for all  $j \in \mathbb{N}$ . Then, for each  $n \in \mathbb{N}$ , and  $\mathbf{v}_j \in \mathbb{R}^n$ ,  $\|\mathbf{v}_j\| \leq 1$  for  $j \in [n]$ ,

$$\sum_{(j,k) \in [n]^2} a_{jk} \langle \mathbf{v}_j, \mathbf{v}_k \rangle \leq K_n \max_{\epsilon_j = \pm 1, j \in [n]} \sum_{(j,k) \in [n]^2} a_{jk} \epsilon_j \epsilon_k, \quad (2.27)$$

where  $\langle \cdot, \cdot \rangle$  denotes the standard dot product in  $\mathbb{R}^n$ , and  $K_n = \mathcal{O}(\log n)$ . Notice that if the "absolute value" is applied to both sides of (2.27), then (by the Grothendieck inequality),  $K_n = \mathcal{O}(1)$ . Notice also that if we take the "decoupled" version of (2.27),

$$\sum_{(j,k) \in [n]^2} a_{jk} \langle \mathbf{v}_j, \mathbf{v}_k \rangle \leq K_n \max_{\epsilon_j = \pm 1, \delta_k = \pm 1} \sum_{j \in [n], k \in [n]} a_{jk} \epsilon_j \delta_k,$$

then (again by the Grothendieck inequality),  $K_n = \mathcal{O}(1)$ .

That  $K_n = \mathcal{O}(\log n)$  is optimal in (2.27) was proved in [1] by sharpening the following result in [24]: for every  $n \in \mathbb{N}$ , there exist vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  in the unit ball of  $\mathbb{R}^n$ , such that whenever *real-valued* functions  $f_1, \dots, f_n$  in  $L^\infty([0, 1])$  verify

$$\langle \mathbf{v}_j, \mathbf{v}_k \rangle = \int_{[0,1]} f_j(t) f_k(t) dt, \quad 1 \leq j < k \leq n \quad (2.28)$$

( $dt = \text{Lebesgue measure}$ ), then

$$\max_{j \in [n]} \|f_j\|_{L^\infty} \geq C(\log n)^{\frac{1}{4}},$$

where  $C$  is an absolute constant.

Notably, if *complex-valued*  $f_1, \dots, f_n$  in  $L^\infty([0, 1])$  are allowed in (2.28), then the opposite phenomenon holds; cf. (1.13), (3.13), and Remark 3.7.i.

### 3. Integral representations: the case of topological domains

**3.1.  $L^2$ -continuous families.** If  $H$  is a Hilbert space, then the Grothendieck inequality, through an application of Proposition 2.4, guarantees that there exists a regular complex measure  $\mu$  on the compact Abelian group

$$\Omega_{B_H} \times \Omega_{B_H} := \{-1, 1\}^{B_H} \times \{-1, 1\}^{B_H},$$

such that

$$\langle \mathbf{u}, \mathbf{v} \rangle_H = \int_{\Omega_{B_H} \times \Omega_{B_H}} r_{\mathbf{u}} \otimes r_{\mathbf{v}} d\mu, \quad (\mathbf{u}, \mathbf{v}) \in B_H \times B_H. \quad (3.1)$$

(Cf. Proposition 1.1.) However, whereas the function on the left side of (3.1) is continuous on  $B_H \times B_H$  (separately in each variable with respect to the weak topology, and jointly with respect to the norm topology), this cannot be independently inferred from the integral representation on the right side of (3.1). To wit, if  $\lambda \in M(\Omega_{B_H} \times \Omega_{B_H})$  is arbitrary, then the function on  $B_H \times B_H$  defined by

$$\hat{\lambda}(r_{\mathbf{u}} \otimes r_{\mathbf{v}}), \quad (\mathbf{u}, \mathbf{v}) \in B_H \times B_H,$$

does not *a priori* “see” any of the structures (linear or topological) in  $H$ . In the case of Theorem 2.7, this is inconsequential: in the definitions of  $\mathcal{V}_2(X \times Y)$ ,  $\tilde{\mathcal{V}}_2(X \times Y)$ , and  $\mathcal{G}_2(X \times Y)$ , the underlying domains  $X$  and  $Y$  are merely sets.

But now suppose that in Questions 2.1, 2.3, and 2.6,  $X$  and  $Y$  are topological spaces, and  $f$  is continuous on  $X \times Y$ . Then, we would want also the representing functions “to see,” somehow, the topologies of their respective domains; specifically, that the integral representations in (2.13) and (2.22) *a priori* determine continuous functions on  $X \times Y$ . In this work we focus on functions that are jointly continuous on  $X \times Y$ . (The “separately continuous” case is briefly discussed in §12.3.)

**Definition 3.1.** Let  $\mathbf{g} = \{g_{\omega}\}_{\omega \in \Omega}$  be a family of scalar-valued functions defined on a topological space  $X$  and indexed by a finite measure space  $e(\Omega, \mu)$ . We say  $\mathbf{g}$  is  $L^2(\mu)$ -continuous if for each  $x \in X$ ,

$$\omega \mapsto g_{\omega}(x), \quad \omega \in \Omega,$$

determines an element of  $L^2(\Omega, \mu)$ , and the resulting map  $\mathbf{g}: X \rightarrow L^2(\Omega, \mu)$  defined by

$$\mathbf{g}(x)(\omega) = g_{\omega}(x), \quad \omega \in \Omega, \quad x \in X,$$

is continuous with respect to the topology on  $X$  (its domain), and the norm topology on  $L^2(\Omega, \mu)$  (its range).

Indeed, if in Questions 2.3 and 2.6, the underlying domains  $X$  and  $Y$  are topological spaces and the families of representing functions are  $L^2$ -continuous, then the subsequent integral representations in (2.13) and (2.22) determine functions that are jointly continuous on  $X \times Y$ .

**Definition 3.2.** Let  $X$  and  $Y$  be topological Hausdorff spaces, and let  $f$  be a scalar-valued function on  $X \times Y$ .

i.  $f \in \tilde{V}_2(X \times Y)$  if there exist a probability space  $(\Omega, \mu)$ , and  $L^2(\mu)$ -continuous families of functions  $\mathbf{g} = \{g_\omega\}_{\omega \in \Omega}$  and  $\mathbf{h} = \{h_\omega\}_{\omega \in \Omega}$  defined on  $X$  and  $Y$  respectively, such that

$$\sup_{x \in X, y \in Y} \|\mathbf{g}(x)\|_{L^\infty(\mu)} \|\mathbf{h}(y)\|_{L^\infty(\mu)} < \infty, \quad (3.2)$$

and

$$f(x, y) = \int_{\Omega} g_\omega(x) h_\omega(y) \mu(d\omega), \quad (x, y) \in X \times Y. \quad (3.3)$$

$\|f\|_{\tilde{V}_2(X \times Y)}$  is the infimum of the left side in (3.2) taken over all indexing spaces  $(\Omega, \mu)$ , and  $L^2(\mu)$ -continuous families of functions that represent  $f$  by (3.3).

ii.  $f \in G_2(X \times Y)$  if there exist a probability space  $(\Omega, \mu)$ , and  $L^2(\mu)$ -continuous families of functions  $\mathbf{g} = \{g_\omega\}_{\omega \in \Omega}$  and  $\mathbf{h} = \{h_\omega\}_{\omega \in \Omega}$  defined on  $X$  and  $Y$  respectively, such that

$$\sup_{x \in X, y \in Y} \|\mathbf{g}(x)\|_{L^2(\mu)} \|\mathbf{h}(y)\|_{L^2(\mu)} < \infty, \quad (3.4)$$

and

$$f(x, y) = \int_{\Omega} g_\omega(x) h_\omega(y) \mu(d\omega), \quad (x, y) \in X \times Y. \quad (3.5)$$

$\|f\|_{G_2(X \times Y)}$  is the infimum of the left side of (3.4) taken over all indexing spaces  $(\Omega, \mu)$ , and  $L^2(\mu)$ -continuous families of functions that represent  $f$  by (3.5).

For the remainder of the section,  $X$  and  $Y$  denote topological Hausdorff spaces. We let  $C_b(X \times Y)$  denote the space of continuous bounded scalar-valued functions on  $X \times Y$  (a Banach algebra with sup-norm and point-wise multiplication on  $X \times Y$ ).

**Proposition 3.3.**  $\|\cdot\|_{\tilde{V}_2}$  and  $\|\cdot\|_{G_2}$  are norms on  $\tilde{V}_2(X \times Y)$  and  $G_2(X \times Y)$ , respectively. With these norms and point-wise multiplication on  $X \times Y$ ,  $\tilde{V}_2(X \times Y)$  and  $G_2(X \times Y)$  are Banach algebras, and

$$\tilde{V}_2(X \times Y) \subset G_2(X \times Y) \subset C_b(X \times Y), \quad (3.6)$$

where inclusions are norm-decreasing.

*Proof.* We only sketch the arguments.

In order to verify that  $\tilde{V}_2(X \times Y)$  and  $G_2(X \times Y)$  are linear spaces, and that the triangle inequality holds for  $\|\cdot\|_{\tilde{V}_2}$  and  $\|\cdot\|_{G_2}$ , note that if functions  $f_1$  and  $f_2$  on  $X \times Y$  are represented, respectively, by  $L^2(\Omega_1, \mu_1)$ - and  $L^2(\Omega_2, \mu_2)$ -continuous families, then  $f_1 + f_2$  can be represented by "sums" of these families, appropriately indexed by the disjoint union of  $(\Omega_1, \mu_1)$  and  $(\Omega_2, \mu_2)$ , properly defined and normalized.



To prove completeness, verify that absolutely summable sequences in  $\tilde{V}_2(X \times Y)$  and  $G_2(X \times Y)$  are summable in their respective spaces, by applying a "countably infinite" version of the argument used to verify the triangle inequality.

To prove that  $\tilde{V}_2(X \times Y)$  and  $G_2(X \times Y)$  are Banach algebras under point-wise multiplication on  $X \times Y$ , in the proof of the triangle inequality replace "sums" of  $L^2$ -continuous families by "products," and replace the disjoint union of  $(\Omega_1, \mu_1)$  and  $(\Omega_2, \mu_2)$  by the product  $(\Omega_1 \times \Omega_2, \mu_1 \times \mu_2)$ .

The constraint in (3.2) is stronger than (3.4), and thus the norm-decreasing left inclusion in (3.6). The  $L^2$ -continuity of the representing families implies (via Cauchy-Schwarz) the norm-decreasing right inclusion. □

**Remark 3.4.** if  $X$  and  $Y$  are discrete, then by Remark 2.5.iv,

$$\tilde{V}_2(X \times Y) = \tilde{\mathcal{V}}_2(X \times Y).$$

If  $X$  and  $Y$  are general topological Hausdorff spaces, then I know only the obvious inclusion

$$\tilde{V}_2(X \times Y) \subset C_b(X \times Y) \cap \tilde{\mathcal{V}}_2(X \times Y).$$

Specifically, if  $X$  and  $Y$  are compact Hausdorff spaces, and  $\tilde{V}(X \times Y)$  is the tilde algebra defined in [40] (and [18, Ch. 11.9]), then (by Proposition 2.4, and the comment on top of p. 26 in [40]) we have

$$\tilde{V}(X \times Y) = C(X \times Y) \cap \tilde{\mathcal{V}}_2(X \times Y),$$

and therefore

$$\tilde{V}_2(X \times Y) \subset \tilde{V}(X \times Y).$$

I do not know whether the reverse inclusion holds.

**3.2. Continuity and *le théorème fondamental*.** The two integral representations in Definition 3.2 are equivalently feasible; that is,

$$G_2(X \times Y) = \tilde{V}_2(X \times Y),$$

which (modulo "best constants") supersedes Theorem 2.7. Specifically, we will prove that there exists  $K > 0$  such that

$$\|f\|_{\tilde{V}_2} \leq K \quad \text{for all } f \in B_{G_2(X \times Y)}. \quad (3.7)$$

To start, observe that  $f \in B_{G_2(X \times Y)}$  means that there exist a Hilbert space  $H$  and maps

$$\begin{aligned} \mathbf{g} : X &\rightarrow B_H, \\ \mathbf{h} : Y &\rightarrow B_H, \end{aligned}$$

continuous with respect to the topologies in  $X$  and  $Y$  and the norm topology on  $B_H$ , such that

$$f(x, y) = \langle \mathbf{g}(x), \mathbf{h}(y) \rangle_H, \quad x \in X, \quad y \in Y.$$

(Cf. (2.24) and (2.25).) Therefore, (3.7) is equivalent to the assertion that the inner product  $\langle \cdot, \cdot \rangle_H$  (viewed as a continuous function on  $B_H \times B_H$ ) is in  $\tilde{V}_2(B_H \times B_H)$ ; that is,

$$\|\langle \cdot, \cdot \rangle_H\|_{\tilde{V}_2(B_H \times B_H)} := K_H < \infty. \quad (3.8)$$

To establish (3.8), we take an infinite-dimensional Hilbert space  $H$ , and fix in it an orthonormal basis  $A$ . We then take  $H$  to be  $l^2(A)$  with the usual dot product, and proceed to construct (algorithmically) an  $L^2$ -continuous family of functions defined on  $l^2(A)$  and indexed by

$$(\Omega_A, \mathbb{P}_A), \quad \mathbb{P}_A = \text{Haar measure}, \quad (3.9)$$

specifically intended as integrands in an integral representation of the dot product.

The main result is stated below. Tools to prove it are collected in §4, and the proof is given in §5.

**Theorem 3.5.** *Let  $A$  be an infinite set. There exists a one-one map*

$$\Phi : l^2(A) \rightarrow L^\infty(\Omega_A, \mathbb{P}_A), \quad (3.10)$$

*which is continuous with respect to the  $l^2(A)$ -norm on its domain and the  $L^2(\Omega_A, \mathbb{P}_A)$ -norm on its range, and has the following properties:*

$$\|\Phi(\mathbf{x})\|_{L^\infty} \leq K \|\mathbf{x}\|_2, \quad \mathbf{x} \in l^2(A), \quad (3.11)$$

*where  $K > 1$  is an absolute constant;*

$$\Phi(c\mathbf{x}) = c\Phi(\mathbf{x}), \quad c \in \mathbb{R}, \quad \mathbf{x} \in l^2(A); \quad (3.12)$$

$$\sum_{\alpha \in A} \mathbf{x}(\alpha) \overline{\mathbf{y}(\alpha)} = \int_{\Omega_A} \Phi(\mathbf{x}) \Phi(\overline{\mathbf{y}}) d\mathbb{P}_A, \quad \mathbf{x} \in l^2(A), \quad \mathbf{y} \in l^2(A). \quad (3.13)$$

**Corollary 3.6.**

(1) *If  $H$  is a Hilbert space with inner product  $\langle \cdot, \cdot \rangle_H$ , then*

$$\|\langle \cdot, \cdot \rangle_H\|_{\tilde{V}_2(B_H \times B_H)} := K_H \leq K^2, \quad (3.14)$$

*where  $K$  is the constant in (3.11).*

(2) *If  $X$  and  $Y$  are topological Hausdorff spaces, then*

$$G_2(X \times Y) = \tilde{V}_2(X \times Y).$$

*Proof.*

(1) Let the indexing space be  $(\Omega_A, \mathbb{P}_A)$ , and take the  $L^2(\Omega_A, \mathbb{P}_A)$ -continuous families of representing functions (in each of the two coordinate) to be  $\Phi = \{\Phi(\cdot)(\omega)\}_{\omega \in \Omega_A}$ .

(2) Suppose  $f$  is a scalar-valued function on  $X \times Y$ , and  $\|f\|_{G_2(X \times Y)} \leq 1$ . Then, there exist an indexing space  $(\Omega, \mu)$ , and  $L^2(\mu)$ -continuous families of representing functions  $\mathbf{g} = \{g_\omega\}_{\omega \in \Omega}$  and  $\mathbf{h} = \{h_\omega\}_{\omega \in \Omega}$ , defined respectively on  $X$  and  $Y$ , such that

$$\sup_{x \in X} \|\mathbf{g}(x)\|_{L^2} \leq 1, \quad \sup_{y \in Y} \|\mathbf{h}(y)\|_{L^2} \leq 1,$$

and

$$f(x, y) = \int_{\Omega} g_\omega(x) h_\omega(y) \mu(d\omega) = \langle \mathbf{g}(x), \overline{\mathbf{h}(y)} \rangle_H, \quad (x, y) \in X \times Y,$$

where  $H = L^2(\Omega, \mu)$ . Fix an orthonormal basis  $A$  for  $H$ , and then let  $\Phi$  be the map in (3.10). Then,  $\Phi \circ \mathbf{g}$  and  $\Phi \circ \bar{\mathbf{h}}$  are  $L^2(\Omega_A, \mathbb{P}_A)$ -continuous families that represent  $f$ , and

$$\sup_{x \in X, y \in Y} \|(\Phi \circ \mathbf{g})(x)\|_{L^\infty(\Omega_A, \mathbb{P}_A)} \|(\Phi \circ \bar{\mathbf{h}})(y)\|_{L^\infty(\Omega_A, \mathbb{P}_A)} \leq K^2,$$

which verifies  $\|f\|_{\tilde{V}_2(X \times Y)} \leq K^2$ . □

### Remark 3.7.

i. It follows from (3.13) and the result in [24] (stated in Remark 2.9.ii) that the map  $\Phi$  does not commute with complex conjugation: the image under  $\Phi$  of  $l_{\mathbb{R}}^2(A)$  ( $= \mathbb{R}^A$  with the Euclidean norm) must contain elements in  $L^\infty(\Omega_A, \mathbb{P}_A)$  with non-zero imaginary parts.

ii. Define

$$\mathcal{K}_{CG} := \sup\{K_H : \text{Hilbert space } H\},$$

where  $K_H$  is the constant in (3.14). Then,

$$K^2 \geq \mathcal{K}_{CG} \geq \mathcal{K}_G, \tag{3.15}$$

where  $\mathcal{K}_G$  is the Grothendieck constant (defined in (2.26)) and  $K$  is the constant in (3.11), with the usual distinction between the real and complex cases. I do not know whether the inequalities in (3.15) are strict. (Cf. Remark 3.4.)

## 4. Tools

The proof of Theorem 3.5 uses harmonic analysis on dyadic groups.

**4.1. The framework.** Given a set  $A$ , we consider the product

$$\Omega_A := \{-1, 1\}^A,$$

and equip it with the usual product topology. The Borel field  $\mathcal{B}_A$  in  $\Omega_A$  is generated by the cylindrical sets

$$C(F; \eta) = \{\omega \in \Omega_A : \omega(\alpha) = \eta(\alpha), \alpha \in F\}, \quad F \subset A, \eta \in \Omega_F.$$

Let  $\mathbb{P}_A$  be the uniform probability measure (infinite product measure) on  $(\Omega_A, \mathcal{B}_A)$ , determined by

$$\mathbb{P}_A(C(F; \eta)) = \left(\frac{1}{2}\right)^{|F|},$$

where  $|F|$  denotes the cardinality of  $F$ . Multiplication in  $\Omega_A$  is defined by

$$(\omega \cdot \omega')(\alpha) = \omega(\alpha)\omega'(\alpha), \quad \omega \in \Omega_A, \omega' \in \Omega_A, \alpha \in A. \quad (4.1)$$

With these structures,  $\Omega_A$  is a compact Abelian group, and  $\mathbb{P}_A$  is its Haar measure.

**4.2. Rademacher and Walsh characters.** Let  $\hat{\Omega}_A$  denote the group of characters of  $\Omega_A$ , wherein group operation is point-wise multiplication of functions. Let  $r_0$  denote the character that is identically 1 on  $\Omega_A$  (multiplicative identity in  $\hat{\Omega}_A$ ). For  $\alpha \in A$ , let  $r_\alpha$  be the  $\alpha^{th}$  coordinate function on  $\Omega_A$ ,

$$r_\alpha(\omega) = \omega(\alpha), \quad \omega \in \Omega_A.$$

The set  $R_A := \{r_\alpha\}_{\alpha \in A}$  is a subset of  $\hat{\Omega}_A$ , and is independent in the following two senses:

- (i) (*Statistical independence*).  $R_A$  is a system of identically distributed independent random variables on the probability space  $(\Omega_A, \mathcal{B}_A, \mathbb{P}_A)$ .
- (ii) (*Algebraic independence*)  $R_A$  is algebraically independent in  $\hat{\Omega}_A$  : for  $F \subset A \cup \{0\}$ ,

$$\prod_{\alpha \in F} r_\alpha = r_0 \quad \Rightarrow \quad F = \{0\}.$$

The system  $R_A$  generates the full character group  $\hat{\Omega}_A$ . Specifically, let  $W_{A,0} = \{r_0\}$ , and

$$W_{A,k} = \left\{ \prod_{\alpha \in F} r_\alpha : F \subset A, |F| = k \right\}, \quad k \in \mathbb{N}.$$

Then,

$$W_A := \bigcup_{k=0}^{\infty} W_{A,k} = \hat{\Omega}_A.$$

We refer to members of  $W_A$  as *Walsh characters*, to members of  $R_A (= W_{A,1})$  as *Rademacher characters*, and to members of  $W_{A,k}$ ,  $k \geq 1$ , as *Walsh characters of order  $k$* .

4.3. **Walsh series.** At the very outset, *Walsh series*

$$S \sim \sum_{w \in W_A} a_w w, \quad (a_w)_{w \in W_A} \in \mathbb{C}^{W_A},$$

are merely formal objects. We write  $\hat{S}(w) = a_w$ , and

$$\text{spect}(S) := \{w \in W_A : \hat{S}(w) \neq 0\}$$

( $\hat{\cdot}$  := the *spectrum* of  $S$ ). Let  $M(\Omega_A)$  denote the space of regular complex measures on  $(\Omega_A, \mathcal{B}_A)$  with the total variation norm. The *Walsh transform* of  $\mu \in M(\Omega_A)$  is

$$\hat{\mu}(w) = \int_{\Omega_A} w(\omega) \mu(d\omega), \quad w \in W_A,$$

and its Walsh series is

$$S[\mu] \sim \sum_{w \in W_A} \hat{\mu}(w) w.$$

If  $f \in L^1(\Omega_A, \mathbb{P}_A)$ , then  $\hat{f} = \widehat{f d\mathbb{P}_A}$ .

If  $f$  is a Walsh polynomial,

$$f = \sum_{w \in F} a_w w, \quad F \subset A, \quad |F| < \infty, \quad (a_w)_{w \in F} \in \mathbb{C}^F,$$

then  $\text{spect}(f) \subset F$ , and  $\hat{f}(w) = a_w$  for  $w \in F$ . Moreover (Parseval's formula),

$$\int_{\Omega_A} f d\mu = \sum_{w \in F} \hat{f}(w) \hat{\mu}(w), \quad \mu \in M(\Omega_A). \quad (4.2)$$

Therefore (because Walsh polynomials are norm-dense in  $C(\Omega_A)$ ), if  $\mu \in M(\Omega_A)$  and  $\hat{\mu} = 0$  on  $\hat{\Omega}_A$ , then  $\mu = 0$ .

If  $\mu \in M(\Omega_A)$ , and

$$\sum_{w \in W_A} |\hat{\mu}(w)|^2 < \infty,$$

then  $\mu \ll \mathbb{P}_A$ , and

$$\frac{d\mu}{d\mathbb{P}_A} \in L^2(\Omega_A, \mathbb{P}_A).$$

In particular,  $W_A$  is a complete orthonormal system in  $L^2(\Omega_A, \mathbb{P}_A)$ , and (Plancherel's formula)

$$\int_{\Omega_A} |f|^2 d\mathbb{P}_A = \sum_{w \in W_A} |\hat{f}(w)|^2, \quad f \in L^2(\Omega_A, \mathbb{P}_A).$$

**4.4. Riesz products.** We define the Riesz product

$$\mathfrak{R}_F(\mathbf{x}) \sim \prod_{\alpha \in F} (r_0 + \mathbf{x}(\alpha)r_\alpha), \quad F \subset A, \quad \mathbf{x} \in \mathbb{C}^F,$$

to be the Walsh series

$$\mathfrak{R}_F(\mathbf{x}) \sim \sum_{k=0}^{\infty} \left( \sum_{w \in W_{F,k}, w=r_{\alpha_1} \cdots r_{\alpha_k}} \mathbf{x}(\alpha_1) \cdots \mathbf{x}(\alpha_k) r_{\alpha_1} \cdots r_{\alpha_k} \right). \quad (4.3)$$

We detail below two classical scenarios (cf. [32], [34]), which play key roles in this work.

**i.** If  $\mathbf{x} \in l_{\mathbb{R}}^{\infty}(F)$  ( $= \mathbb{R}^F$  with the supremum norm) and  $\|\mathbf{x}\|_{\infty} \leq 1$ , then  $\mathfrak{R}_F(\mathbf{x})$  represents a probability measure on  $(\Omega_A, \mathcal{B}_A)$ .

To verify this, let  $f$  be a Walsh polynomial with  $\text{spect}(f) \subset W_B$ , where  $B \subset A$  is finite. Then (by Parseval's formula),

$$\sum_{w \in W_A} \hat{f}(w) \widehat{\mathfrak{R}_F(\mathbf{x})}(w) = \int_{\Omega_A} f(\omega) \mathfrak{R}_{B \cap F}(\mathbf{x})(\omega) \mathbb{P}_A(d\omega). \quad (4.4)$$

Note that  $\mathfrak{R}_{B \cap F}(\mathbf{x})$  is a positive Walsh polynomial, and therefore

$$\|\mathfrak{R}_{B \cap F}\|_{L^1} = \int_{\Omega_A} \mathfrak{R}_{B \cap F}(\mathbf{x}) d\mathbb{P}_A = 1.$$

Then, from (4.4),

$$\left| \sum_{w \in W_A} \hat{f}(w) \widehat{\mathfrak{R}_F(\mathbf{x})}(w) \right| \leq \|f\|_{L^{\infty}}.$$

Therefore, by the Riesz representation theorem and the density of Walsh polynomials in  $C(\Omega_A)$ , (4.3) is the Walsh series of a probability measure  $\mathfrak{R}_F(\mathbf{x})$  on  $(\Omega_A, \mathcal{B}_A)$ .

**ii.** If  $\mathbf{x} \in l_{\mathbb{R}}^2(F)$ , and  $\|\mathbf{x}\|_2 := (\sum_{\alpha \in F} |\mathbf{x}(\alpha)|^2)^{\frac{1}{2}} < \infty$ , then  $\mathfrak{R}_F(i\mathbf{x}) \in L^{\infty}(\Omega_A, \mathbb{P}_A)$ , and

$$\|\mathfrak{R}_F(i\mathbf{x})\|_{L^{\infty}} \leq e^{\frac{\|\mathbf{x}\|_2^2}{2}},$$

where  $i = \sqrt{-1}$ .

Observe that if  $F \subset A$  is finite, then

$$\|\mathfrak{R}_F(i\mathbf{x})\|_{L^{\infty}} = e^{\frac{1}{2} \sum_{\alpha \in F} \log(1+|\mathbf{x}(\alpha)|^2)} \leq e^{\frac{\|\mathbf{x}\|_2^2}{2}}. \quad (4.5)$$

If  $F \subset A$  is arbitrary, and (without loss of generality) countably infinite, then let  $F_n \subset F_{n+1}$ ,

$n = 1, \dots$ , be an increasing sequence of finite sets, such that  $\bigcup_{n=1}^{\infty} F_n = F$ . Then,

$$\lim_{n \rightarrow \infty} \widehat{\mathfrak{R}_{F_n}(i\mathbf{x})}(w) = \widehat{\mathfrak{R}_F(i\mathbf{x})}(w), \quad w \in W_A. \quad (4.6)$$

Therefore, by (4.5) and (4.6), the sequence

$$\mathfrak{R}_{F_n}(i\mathbf{x}), \quad n = 1, \dots,$$

converges in the weak\* topology of  $L^\infty(\Omega_A, \mathbb{P}_A)$  to an element in  $L^\infty(\Omega_A, \mathbb{P}_A)$ , whose Walsh series is

$$\mathfrak{R}_F(i\mathbf{x}) \sim \sum_{k=0}^{\infty} i^k \left( \sum_{\substack{w \in W_{F,k} \\ w=r_{\alpha_1} \cdots r_{\alpha_k}}} \mathbf{x}(\alpha_1) \cdots \mathbf{x}(\alpha_k) r_{\alpha_1} \cdots r_{\alpha_k} \right),$$

and whose  $L^\infty$ -norm is bounded by  $e^{\frac{\|\mathbf{x}\|_2^2}{2}}$ .

**Remark 4.1.** The random variables

$$\mathfrak{R}_{F_n}(i\mathbf{x}) = \prod_{\alpha \in F_n} (r_0 + i\mathbf{x}(\alpha)r_\alpha), \quad n = 1, \dots,$$

form an  $L^\infty$ -bounded martingale sequence, and therefore the martingale convergence theorem implies that the numerical sequence

$$(\mathfrak{R}_{F_n}(i\mathbf{x}))(\omega), \quad n = 1, \dots,$$

converges to  $(\mathfrak{R}_F(i\mathbf{x}))(\omega)$  for almost all  $\omega \in (\Omega_A, \mathbb{P}_A)$ . (E.g., see [43].) Regarding convergence in the  $L^\infty$ -norm, the sequence  $\mathfrak{R}_{F_n}(i\mathbf{x})$ ,  $n = 1, \dots$ , converges uniformly on  $\Omega_A$  if and only if  $\mathbf{x} \in l^1(F)$ , i.e.,  $\sum_{\alpha \in F} |\mathbf{x}(\alpha)| < \infty$ . (See Proposition 6.3.)

For our purposes here, we take the imaginary part of  $\mathfrak{R}_F(i\mathbf{x})$ ,

$$Q_F(\mathbf{x}) := \text{Im} \mathfrak{R}_F(i\mathbf{x}) \sim \sum_{k=0}^{\infty} (-1)^k \left( \sum_{\substack{w \in W_{F,2k+1} \\ w=r_{\alpha_1} \cdots r_{\alpha_{2k+1}}}} \mathbf{x}(\alpha_1) \cdots \mathbf{x}(\alpha_{2k+1}) r_{\alpha_1} \cdots r_{\alpha_{2k+1}} \right), \quad (4.7)$$

and estimate the  $l^2$ -norm of  $\widehat{Q_F(\mathbf{x})}$  restricted to  $R_F^c := W_F \setminus R_F$  as follows: for each  $k \geq 2$ ,

$$\sum_{w \in W_{F,k}, \quad w=r_{\alpha_1} \cdots r_{\alpha_k}} |\mathbf{x}(\alpha_1) \cdots \mathbf{x}(\alpha_k)|^2 \leq \frac{\|\mathbf{x}\|_2^{2k}}{k!}, \quad (4.8)$$

and therefore

$$\|\widehat{Q_F(\mathbf{x})}|_{R_F^c}\|_2 \leq \left( \sum_{k=1}^{\infty} \frac{\|\mathbf{x}\|_2^{2(2k+1)}}{(2k+1)!} \right)^{\frac{1}{2}}. \quad (4.9)$$

We summarize, and record for future use:

**Lemma 4.2.** *Let  $F \subset A$ ,  $\mathbf{x} \in l_{\mathbb{R}}^2(F)$ , and  $\|\mathbf{x}\|_2 = 1$ . Let  $Q_F(\mathbf{x})$  be the imaginary part of the Riesz product  $\Re_F(i\mathbf{x})$ . Then*

$$\text{spect}(Q_F(\mathbf{x})) \subset \bigcup_{k=0}^{\infty} W_{F, 2k+1} ; \quad (4.10)$$

$$\|Q_F(\mathbf{x})\|_{L^\infty} \leq e^{\frac{1}{2}} := \kappa ; \quad (4.11)$$

$$\widehat{Q_F(\mathbf{x})}(r_\alpha) = \mathbf{x}(\alpha), \quad \alpha \in F ; \quad (4.12)$$

$$\|\widehat{Q_F(\mathbf{x})}|_{R_F^c}\|_2 \leq (\sinh(1) - 1)^{\frac{1}{2}} := \delta < 1. \quad (4.13)$$

## 5. Proof of Theorem 3.5

**5.1. Construction of  $\Phi$ .** Let  $\{A_j : j \in \mathbb{N}\}$  be a partition of  $A$  with the property that  $A_j$  and  $A$  have the same cardinality for each  $j \in \mathbb{N}$ . We fix a bijection

$$\tau_0 : A \rightarrow A_1. \quad (5.1)$$

For  $j \in \mathbb{N}$ , denote

$$C_j := \bigcup_{k=1}^{\infty} W_{A_j, 2k+1},$$

and then fix a bijection

$$\tau_j : C_j \rightarrow A_{j+1}.$$

(If  $A$  is infinite, then  $R_A$  and its complement in  $W_A$  have the same cardinality.) For  $\mathbf{x} \in l^2(A)$ , define (cf. (1.15))

$$\sigma \mathbf{x} = \begin{cases} \mathbf{x}/\|\mathbf{x}\|_2 & \text{if } \mathbf{x} \neq \mathbf{0} \\ \mathbf{0} & \text{if } \mathbf{x} = \mathbf{0}. \end{cases} \quad (5.2)$$

We first consider  $\mathbf{x} \in l_{\mathbb{R}}^2(A)$  (= real-valued vectors in  $l^2(A)$ ), and construct  $\Phi(\mathbf{x})$  by recursion. Define  $\mathbf{x}^{(1)} \in l_{\mathbb{R}}^2(A_1)$  by

$$\mathbf{x}^{(1)}(\tau_0 \alpha) = \mathbf{x}(\alpha), \quad \alpha \in A, \quad (5.3)$$



and then define  $\mathbf{x}^{(j)} \in l_{\mathbb{R}}^2(A_j)$ ,  $j = 2, \dots$ , by

$$\mathbf{x}^{(j)}(\tau_{j-1}w) = \|\mathbf{x}^{(j-1)}\|_2 \left( Q_{A_{j-1}}(\sigma \mathbf{x}^{(j-1)}) \right)^\wedge(w), \quad w \in C_{j-1}. \quad (5.4)$$

By (4.10), (4.11), and (4.12) in Lemma 4.2, we have for  $j = 1, \dots$ ,

$$\text{spect}(Q_{A_j}(\sigma \mathbf{x}^{(j)})) \subset W_{A_j}; \quad (5.5)$$

$$\|Q_{A_j}(\sigma \mathbf{x}^{(j)})\|_{L^\infty} \leq \kappa; \quad (5.6)$$

$$\|\mathbf{x}^{(j)}\|_2 \left( Q_{A_j}(\sigma \mathbf{x}^{(j)}) \right)^\wedge(r_\alpha) = \mathbf{x}^{(j)}(\alpha), \quad \alpha \in A_j. \quad (5.7)$$

Also, by (5.3), (5.4), (4.13), and by induction on  $j$ , we obtain

$$\|\mathbf{x}^{(j)}\|_2 \leq \delta^{j-1} \|\mathbf{x}\|_2, \quad j \geq 1. \quad (5.8)$$

Now define

$$\Phi(\mathbf{x}) = \sum_{j=1}^{\infty} i^{j-1} \|\mathbf{x}^{(j)}\|_2 Q_{A_j}(\sigma \mathbf{x}^{(j)}), \quad (5.9)$$

where  $i := \sqrt{-1}$ . The series in (5.9) is absolutely convergent in the  $L^\infty$ -norm. Specifically, by (5.6) and (5.8),

$$\|\Phi(\mathbf{x})\|_{L^\infty} \leq \left( \frac{\kappa}{1-\delta} \right) \|\mathbf{x}\|_2.$$

For  $\mathbf{x} \in l^2(A)$ , write  $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ ,  $\mathbf{u} \in l_{\mathbb{R}}^2(A)$ ,  $\mathbf{v} \in l_{\mathbb{R}}^2(A)$ , and define

$$\Phi(\mathbf{x}) := \Phi(\mathbf{u}) + i\Phi(\mathbf{v}). \quad (5.10)$$

In particular, we have

$$\Phi(\mathbf{x}) := \sum_{j=1}^{\infty} i^{j-1} (\|\mathbf{u}^{(j)}\|_2 Q_{A_j}(\sigma \mathbf{u}^{(j)}) + i\|\mathbf{v}^{(j)}\|_2 Q_{A_j}(\sigma \mathbf{v}^{(j)})), \quad (5.11)$$

and write (for future use)

$$\Phi(\mathbf{x}) = \sum_{j=1}^{\infty} i^{j-1} \Theta_j(\mathbf{x}), \quad (5.12)$$

where

$$\Theta_j(\mathbf{x}) = \|\mathbf{u}^{(j)}\|_2 Q_{A_j}(\sigma \mathbf{u}^{(j)}) + i\|\mathbf{v}^{(j)}\|_2 Q_{A_j}(\sigma \mathbf{v}^{(j)}). \quad (5.13)$$

Then,

$$\|\Theta_j(\mathbf{x})\|_{L^\infty} \leq 2\delta^{j-1} \kappa \|\mathbf{x}\|_2, \quad (5.14)$$

and

$$\|\Phi(\mathbf{x})\|_{L^\infty} \leq \left(\frac{2\kappa}{1-\delta}\right) \|\mathbf{x}\|_2,$$

which proves (3.11) with  $K = \frac{2\kappa}{1-\delta}$ .

If  $\mathbf{x} \neq \mathbf{y}$  then  $\widehat{\Phi(\mathbf{x})} \neq \widehat{\Phi(\mathbf{y})}$ , which verifies that  $\Phi$  is one-one.

**5.2. Homogeneity.** To prove (3.12), it suffices to consider  $\mathbf{x} \in l_{\mathbb{R}}^2(A)$ . By induction on  $j$  (via the recursive definition of  $\mathbf{x}^{(j)}$ ),

$$\mathbf{x}^{(j)} = \|\mathbf{x}\|_2 (\sigma \mathbf{x})^{(j)}, \quad j \geq 1,$$

which implies

$$\Phi(\mathbf{x}) = \|\mathbf{x}\|_2 \Phi(\sigma \mathbf{x}).$$

Therefore, to prove

$$\Phi(c\mathbf{x}) = c\Phi(\mathbf{x}), \quad c \in \mathbb{R}, \quad \mathbf{x} \in l_{\mathbb{R}}^2(A),$$

it suffices to verify it for  $c = -1$ . To this end, note that for  $\mathbf{x} \in l_{\mathbb{R}}^2(A)$ , the Walsh series of  $Q_A(\mathbf{x})$  contains only Walsh characters of odd order, and that

$$\begin{aligned} \widehat{Q_A(-\mathbf{x})}(r_{\alpha_1} \cdots r_{\alpha_{2k+1}}) &= (-1)^{2k+1} \prod_{l=1}^{2k+1} \mathbf{x}(\alpha_l) \\ &= -\widehat{Q_A(\mathbf{x})}(r_{\alpha_1} \cdots r_{\alpha_{2k+1}}). \end{aligned}$$

Therefore, by (5.4),

$$(-\mathbf{x})^{(j)} = -\mathbf{x}^{(j)}, \quad j \geq 1,$$

which, by the definition of  $\Phi$  in (5.9), implies

$$\Phi(-\mathbf{x}) = -\Phi(\mathbf{x}).$$

**5.3. An integral representation of the dot product.** To prove (3.13), it suffices to verify it for  $\mathbf{x} \in l_{\mathbb{R}}^2(A)$  and  $\mathbf{y} \in l_{\mathbb{R}}^2(A)$ . For every integer  $N \geq 1$ ,

$$\begin{aligned} &\int_{\Omega_A} \left( \sum_{j=1}^N i^{j-1} \|\mathbf{x}^{(j)}\|_2 Q_{A_j}(\sigma \mathbf{x}^{(j)}) \right) \left( \sum_{j=1}^N i^{j-1} \|\mathbf{y}^{(j)}\|_2 Q_{A_j}(\sigma \mathbf{y}^{(j)}) \right) d\mathbb{P}_A \quad (5.15) \\ &= \sum_{j=1}^N (-1)^{j-1} \sum_{w \in W_{A_j}} \|\mathbf{x}^{(j)}\|_2 \|\mathbf{y}^{(j)}\|_2 (Q_{A_j}(\sigma \mathbf{x}^{(j)}))^{\wedge}(w) (Q_{A_j}(\sigma \mathbf{y}^{(j)}))^{\wedge}(w) \end{aligned}$$

(by Parseval's formula, by (5.5), and because the  $W_{A_j}$  are pairwise disjoint)

$$= \sum_{j=1}^N (-1)^{j-1} \left( \sum_{\alpha \in A_j} \mathbf{x}^{(j)}(\alpha) \mathbf{y}^{(j)}(\alpha) + \sum_{\alpha \in C_j} \|\mathbf{x}^{(j)}\|_2 \|\mathbf{y}^{(j)}\|_2 (Q_{A_j}(\sigma \mathbf{x}^{(j)}))^{\wedge}(w) (Q_{A_j}(\sigma \mathbf{y}^{(j)}))^{\wedge}(w) \right)$$

(by (5.7))

$$= \sum_{j=1}^N (-1)^{j-1} \left( \sum_{\alpha \in A_j} \mathbf{x}^{(j)}(\alpha) \mathbf{y}^{(j)}(\alpha) + \sum_{\alpha \in A_{j+1}} \mathbf{x}^{(j+1)}(\alpha) \mathbf{y}^{(j+1)}(\alpha) \right)$$

(by (5.4))

$$= \sum_{\alpha \in A} \mathbf{x}(\alpha) \mathbf{y}(\alpha) + (-1)^{N-1} \|\mathbf{x}^{(N)}\|_2 \|\mathbf{y}^{(N)}\|_2 \sum_{\alpha \in C_N} (Q_{A_N}(\sigma \mathbf{x}^{(N)}))^{\wedge}(w) (Q_{A_N}(\sigma \mathbf{y}^{(N)}))^{\wedge}(w) \quad (5.16)$$

(by "telescoping" and by (5.3)).

Letting  $N \rightarrow \infty$  (in (5.15) and (5.16)), we obtain

$$\sum_{\alpha \in A} \mathbf{x}(\alpha) \mathbf{y}(\alpha) = \int_{\Omega_A} \Phi(\mathbf{x}) \Phi(\mathbf{y}) d\mathbb{P}_A.$$

**5.4. ( $l^2 \rightarrow L^2$ )-continuity.** We prove that if  $\mathbf{x}_n \rightarrow \mathbf{x}$  in  $l^2_{\mathbb{R}}(A)$ , then  $\Phi(\mathbf{x}_n) \rightarrow \Phi(\mathbf{x})$  in  $L^2(\Omega_A, \mathbb{P}_A)$ . For  $\mathbf{x} = \mathbf{0}$ , the assertion follows from (3.12). We assume  $\mathbf{x} \neq \mathbf{0}$ , and proceed to verify

$$\sum_{j=1}^{\infty} i^{j-1} (\|\mathbf{x}^{(j)}\|_2 Q_{A_j}(\sigma \mathbf{x}^{(j)}) - \|\mathbf{x}_n^{(j)}\|_2 Q_{A_j}(\sigma \mathbf{x}_n^{(j)})) \xrightarrow{n \rightarrow \infty} \mathbf{0} \quad (5.17)$$

in the  $L^2(\Omega_A, \mathbb{P}_A)$ -norm.

**Lemma 5.1.** *If  $\mathbf{x} \in l^2_{\mathbb{R}}(A)$ ,  $\mathbf{y} \in l^2_{\mathbb{R}}(A)$ ,  $\mathbf{x} \neq \mathbf{0}$ , and  $\mathbf{y} \neq \mathbf{0}$ , then*

$$\|Q_A(\mathbf{x}) - Q_A(\mathbf{y})\|_{L^2(\Omega_A, \mathbb{P}_A)} \leq \|\mathbf{x} - \mathbf{y}\|_2 \sqrt{\sinh(3 \max\{\|\mathbf{x}\|_2^2, \|\mathbf{y}\|_2^2\})}. \quad (5.18)$$

*Proof.* Write  $\|\mathbf{x} - \mathbf{y}\|_2 = \epsilon$ . By Plancherel's theorem and the spectral analysis of Riesz products,

$$\|Q_A(\mathbf{x}) - Q_A(\mathbf{y})\|_{L^2}^2 = \sum_{k=0}^{\infty} \left( \sum_{\substack{w \in W_{A, 2k+1} \\ w = r_{\alpha_1} \cdots r_{\alpha_{2k+1}}} } |\mathbf{x}(\alpha_1) \cdots \mathbf{x}(\alpha_{2k+1}) - \mathbf{y}(\alpha_1) \cdots \mathbf{y}(\alpha_{2k+1})|^2 \right). \quad (5.19)$$

For  $k \geq 1$  (cf. (4.8)),

$$\begin{aligned}
& \sum_{\substack{w \in W_{A,2k+1} \\ w = r_{\alpha_1} \cdots r_{\alpha_{2k+1}}}} |\mathbf{x}(\alpha_1) \cdots \mathbf{x}(\alpha_{2k+1}) - \mathbf{y}(\alpha_1) \cdots \mathbf{y}(\alpha_{2k+1})|^2 \\
& \leq \frac{1}{(2k+1)!} \sum_{\alpha_1, \dots, \alpha_{2k+1}} |\mathbf{x}(\alpha_1) \cdots \mathbf{x}(\alpha_{2k+1}) - \mathbf{y}(\alpha_1) \cdots \mathbf{y}(\alpha_{2k+1})|^2 \\
& \leq \frac{2}{(2k+1)!} \sum_{\alpha_1, \dots, \alpha_{2k+1}} |\mathbf{x}(\alpha_1) \mathbf{x}(\alpha_2) \cdots \mathbf{x}(\alpha_{2k+1}) - \mathbf{y}(\alpha_1) \mathbf{x}(\alpha_2) \cdots \mathbf{x}(\alpha_{2k+1})|^2 \\
& \quad + \frac{2}{(2k+1)!} \sum_{\alpha_1, \dots, \alpha_{2k+1}} |\mathbf{y}(\alpha_1) \mathbf{x}(\alpha_2) \cdots \mathbf{x}(\alpha_{2k+1}) - \mathbf{y}(\alpha_1) \cdots \mathbf{y}(\alpha_{2k+1})|^2 \\
& = \frac{2\epsilon^2}{(2k+1)!} \|\mathbf{x}\|_2^{4k} + \frac{2}{(2k+1)!} \|\mathbf{y}\|_2^2 \sum_{\alpha_2, \dots, \alpha_{2k+1}} |\mathbf{x}(\alpha_2) \cdots \mathbf{x}(\alpha_{2k+1}) - \mathbf{y}(\alpha_2) \cdots \mathbf{y}(\alpha_{2k+1})|^2.
\end{aligned} \tag{5.20}$$

By a recursive application of these estimates ( $2k$  times) to the sum on the right side of (5.20), we obtain that the left side of (5.20) is bounded by

$$\frac{\epsilon^2 2^{2k+1}}{(2k+1)!} \sum_{l=0}^{2k} \|\mathbf{x}\|_2^{2(2k-l)} \|\mathbf{y}\|_2^{2l} < \epsilon^2 \left( \frac{3^{2k+1}}{(2k+1)!} \right) (\max\{\|\mathbf{x}\|_2^2, \|\mathbf{y}\|_2^2\})^{2k+1}. \tag{5.21}$$

An application of (5.21) to each of the summands on the right side of (5.19) implies (5.18).  $\square$

If  $\mathbf{x}_n \rightarrow \mathbf{x}$  in  $l_{\mathbb{R}}^2(A)$ , and  $\mathbf{x} \neq \mathbf{0}$ , then  $\sigma \mathbf{x}_n \rightarrow \sigma \mathbf{x}$  in  $l_{\mathbb{R}}^2(A)$ . Then, by the recursive definition of  $\mathbf{x}^{(j)}$  in (5.4), and by an inductive application of Lemma 4.1, we deduce that for each  $j \geq 1$ ,  $\mathbf{x}_n^{(j)} \rightarrow \mathbf{x}^{(j)}$  in  $l^2(A_j)$ , and therefore,

$$\|\mathbf{x}_n^{(j)}\|_2 Q_{A_j}(\sigma \mathbf{x}_n^{(j)}) \xrightarrow{n \rightarrow \infty} \|\mathbf{x}^{(j)}\|_2 Q_{A_j}(\sigma \mathbf{x}^{(j)}) \tag{5.22}$$

in  $L^2(\Omega_A, \mathbb{P}_A)$ . The assertion in (5.17) now follows from (5.22) and the absolute convergence (in the  $L^\infty$ -norm) of the series in (5.9).

## 6. More about $\Phi$

**6.1. Spectrum.** The construction of  $\Phi(\mathbf{x})$  is based on the spectral analysis of  $Q_{A_j}$  and the recursive definition of  $\mathbf{x}^{(j)} \in l_{\mathbb{R}}^2(A_j)$ . The construction starts with the observation that for infinite  $A$ ,

$$\{B \in 2^A : |B| < \infty\}$$

and  $A$  have the same cardinality. This implies that

$$\text{spect}\Phi := \bigcup_{j,k=1}^{\infty} W_{A_j, 2k-1}$$

and  $A$  have the same cardinality.

If  $\mathbf{x} \in l^2(A)$  has infinite support, then  $\text{spect}\Phi(\mathbf{x})$  is countably infinite. For finitely supported  $\mathbf{x}$ , *except* for two-dimensional vectors  $\mathbf{x}$ , the cardinalities of the supports of  $\mathbf{x}^{(j)}$  grow iterative-exponentially (very fast!) to infinity. To make this precise, denote

$$\nu(\mathbf{x}) := |\{\alpha : \mathbf{x}(\alpha) \neq 0\}|,$$

and let

$$f(k) = 2^k - k - 1, \quad k \in \mathbb{N}.$$

If  $\nu(\mathbf{x}) < \infty$ , then

$$\nu(\mathbf{x}^{(j)}) = f^{j-1}(\nu(\mathbf{x})), \quad j \geq 2,$$

where  $f^{j-1}$  denotes the  $(j-1)^{\text{st}}$  iterate of  $f$ . Therefore,  $\mathbf{x}^{(j)} = \mathbf{0}$  for all  $j \geq 2$  only if  $\nu(\mathbf{x}) \leq 2$ . If  $2 < \nu(\mathbf{x}) < \infty$ , then

$$\nu(\mathbf{x}^{(j)}) = f^{j-1}(\nu(\mathbf{x})) \xrightarrow{j \rightarrow \infty} \infty.$$

That is,

$$\nu(\mathbf{x}) \leq 2 \Rightarrow |\text{spect}(\Phi(\mathbf{x}))| \leq 2,$$

but otherwise,

$$\nu(\mathbf{x}) > 2 \Rightarrow |\text{spect}(\Phi(\mathbf{x}))| = \infty.$$

**6.2.  $(l^2 \rightarrow L^p)$ -continuity.** Lemma 5.1 has the following extension.

**Lemma 6.1.** *If  $\mathbf{x} \in l_{\mathbb{R}}^2(A)$ ,  $\mathbf{y} \in l_{\mathbb{R}}^2(A)$ ,  $0 < \|\mathbf{x}\|_2 \leq 1$ ,  $0 < \|\mathbf{y}\|_2 \leq 1$ , then for all  $p > 2$ , there exist  $K_p > 0$  such that*

$$\|Q_A(\mathbf{x}) - Q_A(\mathbf{y})\|_{L^p(\Omega_A, \mathbb{P}_A)} \leq K_p \|\mathbf{x} - \mathbf{y}\|_2. \quad (6.1)$$

*Proof.* Consider the spectral decomposition

$$Q_A(\mathbf{x}) - Q_A(\mathbf{y}) = \sum_{k=0}^{\infty} (Q_A(\mathbf{x}) - Q_A(\mathbf{y}))_k, \quad (6.2)$$

where

$$\text{spect}((Q_A(\mathbf{x}) - Q_A(\mathbf{y}))_k) \subset W_{A,2k+1}, \quad k = 0, 1, \dots$$

By the estimate in (5.21), and by an application of the  $L^2 - L^p$  inequalities in [10] to  $(Q_A(\mathbf{x}) - Q_A(\mathbf{y}))_k$ , we obtain

$$\|(Q_A(\mathbf{x}) - Q_A(\mathbf{y}))_k\|_{L^p(\Omega_A, \mathbb{P}_A)} \leq \|\mathbf{x} - \mathbf{y}\|_2 \left( \frac{3^{2k+1}}{(2k+1)!} \right)^{\frac{1}{2}} p^{\frac{2k+1}{2}}. \quad (6.3)$$

The estimate in (6.1) now follows from an application of the estimate in (6.3) to each of the summands in (6.2).  $\square$

**Corollary 6.2.** *The maps  $\Theta_j$  defined in (5.13), and (therefore)  $\Phi$ , are  $(l^2 \rightarrow L^p)$ -continuous for every  $p \geq 2$ .*

**6.3.  $(l^2 \rightarrow L^\infty)$ -continuity?** Although bounded in the  $L^\infty(\Omega_A, \mathbb{P}_A)$ -norm,  $\Phi$  is *not* continuous with respect to it. To verify this, we use the following dichotomy.

**Proposition 6.3.** *For  $\mathbf{x} \in l_{\mathbb{R}}^2(A)$ ,*

$$Q_A(\mathbf{x}) \in C(\Omega_A) \iff \mathbf{x} \in l_{\mathbb{R}}^1(A).$$

*Proof.* If  $\mathbf{x} \in l_{\mathbb{R}}^1(A)$ , then

$$Q_A(\mathbf{x}) \sim \sum_{k=0}^{\infty} (-1)^k \left( \sum_{\substack{w \in W_{A,2k+1} \\ w = r_{\alpha_1} \cdots r_{\alpha_{2k+1}}}} \mathbf{x}(\alpha_1) \cdots \mathbf{x}(\alpha_{2k+1}) r_{\alpha_1} \cdots r_{\alpha_{2k+1}} \right),$$

where

$$\sum_{\substack{w \in W_{A,2k+1} \\ w = r_{\alpha_1} \cdots r_{\alpha_{2k+1}}}} |\mathbf{x}(\alpha_1) \cdots \mathbf{x}(\alpha_{2k+1})| \leq \frac{\|\mathbf{x}\|_1^{2k+1}}{(2k+1)!}, \quad k \geq 0$$

(cf. (4.7) and (4.8)), which implies

$$\sum_{w \in W_A} |\widehat{Q_A(\mathbf{x})}(w)| \leq \sinh(\|\mathbf{x}\|_1), \quad (6.4)$$

and thus  $Q_A(\mathbf{x}) \in C(\Omega_A)$ .

Now suppose  $Q_A(\mathbf{x}) \in C(\Omega_A)$ , and without loss of generality assume  $|\mathbf{x}(\alpha)| < 1$  for all  $\alpha \in A$ . Let  $E_n$ ,  $n = 1, \dots$ , be a sequence of finite subsets of  $A$ , monotonically increasing

to  $E := \text{support}(\mathbf{x})$ . Then,  $Q_A(\mathbf{x}) = Q_E(\mathbf{x})$ , and (e.g., by [7, Corollary VII.9])  $Q_{E_n}(\mathbf{x}) \xrightarrow{n \rightarrow \infty} Q_E(\mathbf{x})$  uniformly on  $\Omega_A$ . Therefore,

$$\sum_{\alpha \in E_n} \log(1 + \mathbf{x}(\alpha)r_\alpha) \xrightarrow{n \rightarrow \infty} \sum_{\alpha \in E} \log(1 + \mathbf{x}(\alpha)r_\alpha)$$

uniformly on  $\Omega_A$ , which implies

$$\sum_{\alpha \in E_n} \mathbf{x}(\alpha)r_\alpha \xrightarrow{n \rightarrow \infty} \sum_{\alpha \in E} \mathbf{x}(\alpha)r_\alpha$$

uniformly on  $\Omega_A$ , and hence  $\sum_{\alpha \in A} |\mathbf{x}(\alpha)| < \infty$ . □

**Corollary 6.4.** *For  $\mathbf{x} \in l^2(A)$ ,*

$$\Phi(\mathbf{x}) \in C(\Omega_A) \iff \mathbf{x} \in l^1(A).$$

*Proof.* If  $\mathbf{x} \in l^1_{\mathbb{R}}(A)$ , then by the recursive definition of  $\mathbf{x}^{(j)}$  and (6.4),

$$(Q_{A_j}(\sigma \mathbf{x}^{(j)}))^{\wedge} \in l^1(W_{A_j}),$$

and therefore

$$Q_{A_j}(\sigma \mathbf{x}^{(j)}) \in C(\Omega_A), \quad j \geq 1.$$

Therefore,  $\Phi(\mathbf{x}) \in C(\Omega_A)$ .

Conversely, if  $\Phi(\mathbf{x}) \in C(\Omega_A)$ , then  $Q_{A_1}(\sigma \mathbf{x}) \in C(\Omega_{A_1})$  (because the  $W_{A_j}$  are independent), and therefore  $\mathbf{x} \in l^1(A)$  (by Proposition 6.3). □

**Corollary 6.5.**  *$\Phi$  is not  $(l^2 \rightarrow L^\infty)$ -continuous.*

*Proof.* Every  $\mathbf{x} \in l^2(A)$  is an  $l^2$ -limit of finitely supported members of  $l^2(A)$ . But if  $\mathbf{x}$  has finite support, then  $\Phi(\mathbf{x}) \in C(\Omega_A)$ , and therefore,  $(l^2 \rightarrow L^\infty)$ -continuity of  $\Phi$  would contradict Corollary 6.4. □

**6.4. Linearization.** The  $(l^2 \rightarrow L^2)$ -continuity of  $\Phi$  implies that its image

$$\Phi[l^2] := \{\Phi(\mathbf{x}) : \mathbf{x} \in l^2(A)\}$$

is norm-closed in  $L^2(\Omega_A, \mathbb{P}_A)$ , and (therefore) also norm-closed in  $L^\infty(\Omega_A, \mathbb{P}_A)$ . However, the map  $\Phi$  is obviously not linear. We consider its linear span, with the following equivalence relation in it: for  $f$  and  $g$  in  $\text{span}(\Phi[l^2])$ ,

$$f \equiv g \iff \int_{\Omega_A} f \Phi(\mathbf{z}) d\mathbb{P}_A = \int_{\Omega_A} g \Phi(\mathbf{z}) d\mathbb{P}_A, \quad \forall \mathbf{z} \in l^2(A). \quad (6.5)$$

By (3.13) (in Theorem 3.5),

$$f \equiv g \iff \hat{f}|_{R_{A_1}} = \hat{g}|_{R_{A_1}}.$$

In particular, if  $f \in \text{span}(\Phi[l^2])$ , then there exists a unique  $\mathbf{x} \in l^2(A)$  such that  $f \equiv \Phi(\mathbf{x})$ . We take the resulting quotient space

$$\begin{aligned} H &:= \text{span}(\Phi[l^2]) / \equiv \\ &= \text{span}(\Phi[l^2]) / \{f \in \text{span}(\Phi[l^2]) : \hat{f}|_{R_{A_1}} = 0\}, \end{aligned}$$

together with the quotient map

$$\tilde{\Phi} : l^2(A) \rightarrow H,$$

i.e.,  $\tilde{\Phi}(\mathbf{x})$  is the equivalence class in  $H$  represented by  $\Phi(\mathbf{x})$ . The dot product  $\langle \cdot, \cdot \rangle_{l^2(A)}$  in  $l^2(A)$  becomes the inner product  $\gamma$  in  $H$ ,

$$\gamma(f, g) := \int_{\Omega_A} \Phi(\mathbf{x}_f) \Phi(\bar{\mathbf{x}}_g) d\mathbb{P}_A, \quad f \in H, \quad g \in H, \quad (6.6)$$

$$= \langle \mathbf{x}_f, \mathbf{x}_g \rangle_{l^2(A)}, \quad (6.7)$$

where  $\mathbf{x}_f \in l^2(A)$  is given by

$$\mathbf{x}_f(\alpha) = \hat{f}(r_{\tau_0 \alpha}), \quad \alpha \in A,$$

and  $\tau_0$  is defined in (5.1). Equipped with  $\gamma$ , the resulting Hilbert space  $H$  is unitarily equivalent (via  $\tilde{\Phi}$ ) to  $l^2(A)$ : for  $\mathbf{x}$  and  $\mathbf{y}$  in  $l^2(A)$ ,

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle &= \int_{\Omega_A} \Phi(\mathbf{x}) \Phi(\bar{\mathbf{y}}) d\mathbb{P}_A \\ &\quad (\text{by (3.13)}) \end{aligned} \quad (6.8)$$

$$\begin{aligned} &= \gamma(\tilde{\Phi}(\mathbf{x}), \tilde{\Phi}(\mathbf{y})) \\ &\quad (\text{by (6.6)}). \end{aligned}$$

Moreover, the Hilbert space norm is equivalent (also via  $\tilde{\Phi}$ ) to the quotient  $L^\infty$ -norm,

$$\|f\|_{\tilde{L}^\infty} := \inf\{\|g\|_{L^\infty} : g \in \text{span}(\Phi[l^2]), \quad g \equiv f\}, \quad f \in H. \quad (6.9)$$

In summary, the map  $\tilde{\Phi} : l^2(A) \rightarrow H$  is linear, and satisfies (3.13) in the sense that for  $\mathbf{x} \in l^2(A)$  and  $\mathbf{y} \in l^2(A)$ ,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{\Omega_A} \tilde{\Phi}(\mathbf{x}) \tilde{\Phi}(\bar{\mathbf{y}}) d\mathbb{P}_A, \quad (6.10)$$

where integrands are members of the respective equivalence classes  $\tilde{\Phi}(\mathbf{x})$  and  $\tilde{\Phi}(\bar{\mathbf{y}})$ , and the integral above is well defined by (6.5). Observe that  $\tilde{\Phi}$  does not commute with complex conjugation; see Remark 3.7.i. Following (3.11), we also have

$$\|\tilde{\Phi}(\mathbf{x})\|_{\tilde{L}^\infty} \leq K \|\mathbf{x}\|_2, \quad \mathbf{x} \in l^2(A). \quad (6.11)$$

Finally, note that  $\tilde{\Phi}$  is continuous with respect to the norm topologies in the Hilbert spaces



$l^2(A)$  and  $H$  (via the  $(l^2 \rightarrow L^2)$ -continuity of  $\Phi$ ), and continuous also with respect to their weak topologies (via the unitary equivalence in (6.8)).

## 7. Integrability

We will now view  $\Phi$  as an  $L^\infty(\Omega_A, \mathbb{P}_A)$ -valued function on the unit ball  $B_{l^2} := B_{l^2(A)}$  with the weak topology on it. Although the  $L^\infty$ -valued  $\Phi$  is not weakly continuous on  $B_{l^2}$  (e.g., §12.3), it is weak\*-measurable, i.e., for every  $f \in L^1(\Omega_A, \mathbb{P}_A)$ ,

$$\int_{\omega \in \Omega_A} \Phi(\mathbf{x})(\omega) f(\omega) \mathbb{P}_A(d\omega), \quad \mathbf{x} \in B_{l^2},$$

determines a scalar-valued  $\mathcal{B}_{B_{l^2}}$ -measurable function on  $B_{l^2}$ , where  $\mathcal{B}_{B_{l^2}}$  is the Borel field in  $B_{l^2}$  generated by the weak topology on  $l^2(A)$ . Next we consider integration over  $B_{l^2}$ .

**7.1. Integrability of  $\Phi$ .** We let  $M(B_{l^2})$  be space of complex measures on  $\mathcal{B}_{B_{l^2}}$ , and proceed to define

$$\int_{B_{l^2}} \Phi(\mathbf{x}) \mu(d\mathbf{x}), \quad \mu \in M(B_{l^2}), \quad (7.1)$$

as an  $L^\infty$ -valued integral, specifically as a weak\* limit in  $L^\infty(\Omega_A, \mathbb{P}_A)$ .

For some measures (e.g., discrete measures, or measures supported on  $l^1(A)$ ), the integrals above can be evaluated numerically, point-wise almost surely on  $(\Omega_A, \mathbb{P})$ . But otherwise, for arbitrary  $\mu \in M(B_{l^2})$ , these integrals can be defined only "weakly." Indeed, fixing  $\mathbf{x} \in B_{l^2(A)}$ , we have an exceptional  $\mathbb{P}_A$ -null set  $E_{\mathbf{x}} \subset \Omega_A$  on whose complement the scalar-valued function  $\Phi(\mathbf{x})(\cdot)$  is uniformly bounded and measurable. But when varying  $\mathbf{x}$ , we lose control of the "size" of  $\bigcup_{\mathbf{x} \in B_{l^2}} E_{\mathbf{x}}$ , and, in particular, cannot assert that for almost all  $\omega \in (\Omega_A, \mathbb{P}_A)$ , the integrands  $\Phi(\cdot)(\omega)$  in (7.1) are  $\mathcal{B}_{B_{l^2}}$ -measurable.

**Proposition 7.1.** *For  $\mu \in M(B_{l^2})$ , the Walsh series*

$$\int_{B_{l^2}} \Phi(\mathbf{x}) \mu(d\mathbf{x}) := \sum_{j=1}^{\infty} i^{j-1} \left( \sum_{w \in W_{A_j}} \left( \int_{B_{l^2}} \widehat{\Theta_j(\mathbf{x})}(w) \mu(d\mathbf{x}) \right) w \right) \quad (7.2)$$

*represents an element of  $L^\infty(\Omega_A, \mathbb{P}_A)$ , such that*

$$\left\| \int_{B_{l^2}} \Phi(\mathbf{x}) \mu(d\mathbf{x}) \right\|_{L^\infty} \leq \left( \frac{2\kappa}{1-\delta} \right) \|\mu\|_M, \quad (7.3)$$

*where  $\Theta_j$  is given in (5.13),  $\|\cdot\|_M$  is the usual total variation norm, and  $\kappa$  and  $\delta$  are the constants in (4.11) and (4.13).*

Proposition 7.1 is a consequence of the following.

**Lemma 7.2.** *For  $\mu \in M(B_{l^2})$  and  $j \in \mathbb{N}$ , the Walsh series*

$$\int_{B_{l^2}} \Theta_j(\mathbf{x}) \mu(d\mathbf{x}) := \sum_{w \in W_{A_j}} \left( \int_{B_{l^2}} \widehat{\Theta_j(\mathbf{x})}(w) \mu(d\mathbf{x}) \right) w \quad (7.4)$$

*represents an element of  $L^\infty(\Omega_A, \mathbb{P}_A)$ , and*

$$\left\| \int_{B_{l^2}} \Theta_j(\mathbf{x}) \mu(d\mathbf{x}) \right\|_{L^\infty} \leq 2\kappa \delta^{j-1} \|\mu\|_M. \quad (7.5)$$

*Proof.* To check that the integrals on the right side of (7.4) are well defined, first observe that the function

$$\mathbf{x} \mapsto \|\mathbf{x}\|_2, \quad \mathbf{x} \in B_{l^2},$$

is weakly lower semicontinuous and hence  $\mathcal{B}_{l^2}$ -measurable, and then conclude (by induction) that for all  $j \in \mathbb{N}$  and  $w \in W_{A_j}$ ,

$$\mathbf{x} \mapsto \widehat{\Theta_j(\mathbf{x})}(w), \quad \mathbf{x} \in B_{l^2_{\mathbb{R}}}, \quad (7.6)$$

are bounded  $\mathcal{B}_{l^2_{\mathbb{R}}}$ -measurable functions.

For  $n = 1, \dots$ , define

$$\rho_n \mathbf{x} = \mathbf{1}_{\{\|\mathbf{x}\|_2 \geq \frac{1}{n}\}} \mathbf{x}, \quad \mathbf{x} \in B_{l^2}, \quad (7.7)$$

and note that the support of  $\rho_n \mathbf{x}$  is finite. Then for each  $n$ , consider the Walsh polynomial

$$\int_{B_{l^2}} \Theta_j(\rho_n \mathbf{x}) \mu(dx) = \sum_{w \in W_{A_j}} \left( \int_{B_{l^2}} (\Theta_j(\rho_n \mathbf{x}))^\wedge(w) \mu(d\mathbf{x}) \right) w, \quad (7.8)$$

and by (5.14) estimate

$$\left\| \int_{B_{l^2}} \Theta_j(\rho_n \mathbf{x}) \mu(dx) \right\|_{L^\infty} \leq 2\kappa \delta^{j-1} \|\mu\|_M. \quad (7.9)$$

Letting  $n \rightarrow \infty$ , we have  $\rho_n \mathbf{x} \rightarrow \mathbf{x}$  in the  $l^2$ -norm, and therefore (by (5.22)) for every  $w \in W_{A_j}$ ,

$$\int_{B_{l^2}} (\Theta_j(\rho_n \mathbf{x}))^\wedge(w) \mu(d\mathbf{x}) \xrightarrow{n \rightarrow \infty} \int_{B_{l^2}} \widehat{\Theta_j(\mathbf{x})}(w) \mu(d\mathbf{x}). \quad (7.10)$$

Finally, we deduce from (7.9) and (7.10) that the polynomials in (7.8) converge in the weak\* topology of  $L^\infty(\Omega_A, \mathbb{P}_A)$  to an element in  $L^\infty(\Omega_A, \mathbb{P}_A)$ , whose Walsh series is the right side of (7.4).  $\square$

**7.2. Integrability of  $\Phi \otimes \Phi$ .** Next we consider the two-variable case, checking for integrability of  $\Phi \otimes \Phi$ , where

$$(\Phi \otimes \Phi)(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x})\Phi(\mathbf{y}), \quad (\mathbf{x}, \mathbf{y}) \in B_{l^2} \times B_{l^2}.$$

Mimicking integration with respect to complex measures (Proposition 7.1), integration over  $B_{l^2} \times B_{l^2}$  is performed, analogously, with respect to  $\mathcal{F}_2$ -measures (known also as *bimeasures*): scalar-valued set-functions  $\mu$  defined on "rectangles"  $(E, F) \in \mathcal{B}_{l^2} \times \mathcal{B}_{l^2}$ , such that for every  $E \in \mathcal{B}_{l^2}$  and  $F \in \mathcal{B}_{l^2}$ ,

$$\mu(E, \cdot) \quad \text{and} \quad \mu(\cdot, F)$$

are, respectively, complex measures on  $\mathcal{B}_{l^2}$ . The space of all such set-functions is denoted by  $\mathcal{F}_2(\mathcal{B}_{l^2}, \mathcal{B}_{l^2})$ , and is a generalization of  $\mathcal{F}_2(X \times Y)$  defined in (2.9). Given bounded scalar-valued measurable functions  $f$  and  $g$  on  $B_{l^2}$ , and  $\mu \in \mathcal{F}_2(\mathcal{B}_{l^2}, \mathcal{B}_{l^2})$ , we have a Lebesgue-type bilinear integral

$$\int_{B_{l^2} \times B_{l^2}} f(\mathbf{x})g(\mathbf{y})\mu(d\mathbf{x}, d\mathbf{y}), \quad (7.11)$$

which is computed iteratively: first with respect to  $\mathbf{x}$  and then with respect to  $\mathbf{y}$ , or vice versa. The  $\mathcal{F}_2$ -norm of  $\mu$  is

$$\|\mu\|_{\mathcal{F}_2} := \sup \left\{ \left| \int_{B_{l^2} \times B_{l^2}} f(\mathbf{x})g(\mathbf{y})\mu(d\mathbf{x}, d\mathbf{y}) \right| : \|f\|_{\mathcal{L}^\infty} \leq 1, \|g\|_{\mathcal{L}^\infty} \leq 1 \right\}, \quad (7.12)$$

where  $f$  and  $g$  in (7.12) are measurable functions on  $B_{l^2}$ , and  $\|\cdot\|_{\mathcal{L}^\infty}$  denotes supremum over  $B_{l^2}$ . ( $\mathcal{F}$  is for Maurice Fréchet, first to study bilinear analogs of measures, and in particular, first to construct integrals of the type in (7.11); see [17], [7, Ch. VI], and also the discussion leading to Lemma 2.2.)

To verify the two-variable analog of Proposition 7.1, we use

**Theorem 7.3** (The Grothendieck factorization theorem). *If  $\mu \in \mathcal{F}_2(\mathcal{B}_{l^2}, \mathcal{B}_{l^2})$ , then there exist probability measures  $\nu_1$  and  $\nu_2$  on  $\mathcal{B}_{l^2}$  such that for all bounded measurable functions  $f$  and  $g$  on  $B_{l^2}$ ,*

$$\left| \int_{B_{l^2} \times B_{l^2}} f(\mathbf{x})g(\mathbf{y})\mu(d\mathbf{x}, d\mathbf{y}) \right| \leq \mathcal{K}_G \|\mu\|_{\mathcal{F}_2} \|f\|_{L^2(\nu_1)} \|g\|_{L^2(\nu_2)}, \quad (7.13)$$

where  $\mathcal{K}_G$  is the constant in (2.26).

Theorem 7.3 is an assertion equivalent to the Grothendieck inequality: either statement is derivable from the other, with the same constant  $\mathcal{K}_G$  in both; e.g., see [7, Ch. V].

Fixing an arbitrary  $\mu \in \mathcal{F}_2(\mathcal{B}_{l^2}, \mathcal{B}_{l^2})$ , we write (formally)

$$\begin{aligned} & \int_{B_{l^2} \times B_{l^2}} \Phi(\mathbf{x})\Phi(\mathbf{y})\mu(d\mathbf{x}, d\mathbf{y}) \\ &:= \sum_{\substack{j,k=1 \\ j \neq k}}^{\infty} i^{(j+k-2)} \int_{B_{l^2} \times B_{l^2}} \Theta_j(\mathbf{x})\Theta_k(\mathbf{y})\mu(d\mathbf{x}, d\mathbf{y}) \\ & \quad + \sum_{k=1}^{\infty} (-1)^{k-1} \int_{B_{l^2} \times B_{l^2}} \Theta_k(\mathbf{x})\Theta_k(\mathbf{y})\mu(d\mathbf{x}, d\mathbf{y}), \end{aligned} \tag{7.14}$$

and proceed to verify that the summands in the two sums on the right side of (7.14) are elements in  $L^\infty(\Omega_A, \mathbb{P}_A)$ , and that the sums converge in the  $L^\infty$ -norm.

The first sum on the right side of (7.14) is handled by the following.

**Lemma 7.4.** *For  $\mu \in \mathcal{F}_2(\mathcal{B}_{l^2}, \mathcal{B}_{l^2})$  and positive integers  $j \neq k$ , the Walsh series*

$$\begin{aligned} & \int_{B_{l^2} \times B_{l^2}} \Theta_j(\mathbf{x})\Theta_k(\mathbf{y})\mu(d\mathbf{x}, d\mathbf{y}) \\ &:= \sum_{w_1 \in W_{A_j}, w_2 \in W_{A_k}} \left( \int_{B_{l^2} \times B_{l^2}} \widehat{\Theta_j(\mathbf{x})(w_1)} \widehat{\Theta_k(\mathbf{y})(w_2)} \mu(d\mathbf{x}, d\mathbf{y}) \right) w_1 w_2 \end{aligned} \tag{7.15}$$

*represents an element of  $L^\infty(\Omega_A, \mathbb{P}_A)$ , and*

$$\left\| \int_{B_{l^2} \times B_{l^2}} \Theta_j(\mathbf{x})\Theta_k(\mathbf{y})\mu(d\mathbf{x}, d\mathbf{y}) \right\|_{L^\infty} \leq 4\kappa^2 \delta^{j+k-2} \|\mu\|_{\mathcal{F}_2}. \tag{7.16}$$

*Sketch of proof.* For  $w_1 \in W_{A_j}$ ,  $w'_1 \in W_{A_j}$ , and  $w_2 \in W_{A_k}$ ,  $w'_2 \in W_{A_k}$ ,

$$w_1 w_2 = w'_1 w'_2 \implies w_1 = w'_1, \quad w_2 = w'_2,$$

because  $W_{A_j}$  and  $W_{A_k}$  are mutually independent. Therefore, the series on the right side of (7.15) is a *bona fide* Walsh series.

The proof of Lemma 7.4 is nearly identical to the proof Lemma 7.2: the complex measure  $\mu$  and the integrals with respect to it in Lemma 7.2 are replaced by the  $\mathcal{F}_2$ -measure  $\mu$  and bilinear integrals with respect to it (as per (7.11) and (7.12)). Details are omitted.  $\square$

The second sum on the right side of (7.14) requires an intervention of Theorem 7.3.

**Lemma 7.5.** *For  $\mu \in \mathcal{F}_2(\mathcal{B}_{l^2}, \mathcal{B}_{l^2})$  and  $k \in \mathbb{N}$ , the Walsh series*

$$\begin{aligned} & \int_{B_{l^2} \times B_{l^2}} \Theta_k(\mathbf{x}) \Theta_k(\mathbf{y}) \mu(d\mathbf{x}, d\mathbf{y}) \\ &:= \sum_{w \in W_{A_k}} \left( \sum_{\substack{w_1 \in W_{A_k}, w_2 \in W_{A_k} \\ w_1 w_2 = w}} \int_{B_{l^2} \times B_{l^2}} \widehat{\Theta_k(\mathbf{x})(w_1)} \widehat{\Theta_k(\mathbf{y})(w_2)} \mu(d\mathbf{x}, d\mathbf{y}) \right) w \end{aligned} \quad (7.17)$$

*represents an element of  $L^\infty(\Omega_A, \mathbb{P}_A)$ , and*

$$\left\| \int_{B_{l^2} \times B_{l^2}} \Theta_k(\mathbf{x}) \Theta_k(\mathbf{y}) \mu(d\mathbf{x}, d\mathbf{y}) \right\|_{L^\infty} \leq 4\kappa^2 \delta^{2k-2} \|\mu\|_{\mathcal{F}_2}. \quad (7.18)$$

*Proof.* We first verify that for fixed  $w \in W_{A_k}$ , the sum on the right side of (7.15)

$$\begin{aligned} & \sum_{\substack{w_1 \in W_{A_k}, w_2 \in W_{A_k} \\ w_1 w_2 = w}} \left( \int_{B_{l^2} \times B_{l^2}} \widehat{\Theta_k(\mathbf{x})(w_1)} \widehat{\Theta_k(\mathbf{y})(w_2)} \mu(d\mathbf{x}, d\mathbf{y}) \right) w_1 w_2 \\ &= \sum_{w_1 \in W_{A_k}} \left( \int_{B_{l^2} \times B_{l^2}} \widehat{\Theta_k(\mathbf{x})(w_1)} \widehat{\Theta_k(\mathbf{y})(w_1 w)} \mu(d\mathbf{x}, d\mathbf{y}) \right) w_1 \end{aligned} \quad (7.19)$$

converges absolutely. To this end, let  $\nu_1$  and  $\nu_2$  be probability measures on  $\mathcal{B}_{l^2}$  associated with  $\mu$ , as per Theorem 7.3. Then by (7.13), for every  $w_1 \in W_{A_k}$  we estimate

$$\begin{aligned} & \left| \int_{B_{l^2} \times B_{l^2}} \widehat{\Theta_k(\mathbf{x})(w_1)} \widehat{\Theta_k(\mathbf{y})(w_1 w)} \mu(d\mathbf{x}, d\mathbf{y}) \right| \\ & \leq \mathcal{K}_G \|\mu\|_{\mathcal{F}_2} \left( \int_{B_{l^2}} |\widehat{\Theta_k(\mathbf{x})(w_1)}|^2 \nu_1(d\mathbf{x}) \right)^{\frac{1}{2}} \left( \int_{B_{l^2}} |\widehat{\Theta_k(\mathbf{y})(w_1 w)}|^2 \nu_2(d\mathbf{y}) \right)^{\frac{1}{2}}. \end{aligned} \quad (7.20)$$

Applying the estimate in (7.20) to the summands on the right side of (7.19), and then applying Cauchy-Schwarz, Plancherel, and (5.14), we obtain

$$\begin{aligned}
& \sum_{w_1 \in W_{A_k}} \left| \int_{B_{l^2} \times B_{l^2}} \widehat{\Theta_k(\mathbf{x})(w_1)} \widehat{\Theta_k(\mathbf{y})(w_1 w)} \mu(d\mathbf{x}, d\mathbf{y}) \right| \\
& \leq \mathcal{K}_G \|\mu\|_{\mathcal{F}_2} \left( \sum_{w_1 \in W_{A_k}} \int_{B_{l^2}} |\widehat{\Theta_k(\mathbf{x})(w_1)}|^2 \nu_1(d\mathbf{x}) \right)^{\frac{1}{2}} \left( \sum_{w_1 \in W_{A_k}} \int_{B_{l^2}} |\widehat{\Theta_k(\mathbf{y})(w_1 w)}|^2 \nu_2(d\mathbf{y}) \right)^{\frac{1}{2}} \\
& = \mathcal{K}_G \|\mu\|_{\mathcal{F}_2} \left( \int_{B_{l^2}} \left( \sum_{w_1 \in W_{A_k}} |\widehat{\Theta_k(\mathbf{x})(w_1)}|^2 \right) \nu_1(d\mathbf{x}) \right)^{\frac{1}{2}} \left( \int_{B_{l^2}} \left( \sum_{w_1 \in W_{A_k}} |\widehat{\Theta_k(\mathbf{y})(w_1 w)}|^2 \right) \nu_1(d\mathbf{y}) \right)^{\frac{1}{2}} \\
& \leq \mathcal{K}_G \|\mu\|_{\mathcal{F}_2} \left( \int_{B_{l^2}} \|\Theta_k(\mathbf{x})\|_{L^\infty(\Omega_A, \mathbb{P}_A)}^2 \nu_1(d\mathbf{x}) \right)^{\frac{1}{2}} \left( \int_{B_{l^2}} \|\Theta_k(\mathbf{y})\|_{L^\infty(\Omega_A, \mathbb{P}_A)}^2 \nu_1(d\mathbf{y}) \right)^{\frac{1}{2}} \\
& \leq 4\kappa^2 \delta^{2(k-1)} \mathcal{K}_G \|\mu\|_{\mathcal{F}_2},
\end{aligned} \tag{7.21}$$

which verifies that (7.19) converges absolutely.

We now proceed as in the proofs of Lemmas 7.2 and 7.4. For  $w \in W_{A_k}$   $w_1 \in W_{A_k}$ ,

$$\begin{aligned}
& \int_{B_{l^2} \times B_{l^2}} (\Theta_k(\rho_n \mathbf{x}))^\wedge(w_1) (\Theta_k(\rho_n \mathbf{y}))^\wedge(w_1 w) \mu(d\mathbf{x}, d\mathbf{y}) \\
& \xrightarrow{n \rightarrow \infty} \int_{B_{l^2} \times B_{l^2}} \widehat{\Theta_k(\mathbf{x})(w_1)} \widehat{\Theta_k(\mathbf{y})(w_1 w)} \mu(d\mathbf{x}, d\mathbf{y}),
\end{aligned} \tag{7.22}$$

where  $\rho_n$  is defined in (7.7); cf. (7.10). Therefore, via the absolute convergence of (7.19), we obtain

$$\begin{aligned}
& \left( \int_{B_{l^2} \times B_{l^2}} \Theta_k(\rho_n \mathbf{x}) \Theta_k(\rho_n \mathbf{y}) \mu(d\mathbf{x}, d\mathbf{y}) \right)^\wedge(w) \\
& \xrightarrow{n \rightarrow \infty} \sum_{\substack{w_1 \in W_{A_k}, w_2 \in W_{A_k} \\ w_1 w_2 = w}} \int_{B_{l^2} \times B_{l^2}} \widehat{\Theta_k(\mathbf{x})(w_1)} \widehat{\Theta_k(\mathbf{y})(w_2)} \mu(d\mathbf{x}, d\mathbf{y}) \\
& = \left( \int_{B_{l^2} \times B_{l^2}} \Theta_k(\mathbf{x}) \Theta_k(\mathbf{y}) \mu(d\mathbf{x}, d\mathbf{y}) \right)^\wedge(w).
\end{aligned} \tag{7.23}$$

Finally, by combining (7.23) with the estimate

$$\left\| \int_{B_{l^2} \times B_{l^2}} \Theta_k(\rho_n \mathbf{x}) \Theta_k(\rho_n \mathbf{y}) \mu(d\mathbf{x}, d\mathbf{y}) \right\|_{L^\infty} \leq 4\kappa^2 \delta^{2(k-1)} \|\mu\|_{\mathcal{F}_2} \quad (7.24)$$

(cf. (7.9) and (7.10)), we deduce that the sequence of Walsh polynomials

$$\int_{B_{l^2} \times B_{l^2}} \Theta_k(\rho_n \mathbf{x}) \Theta_k(\rho_n \mathbf{y}) \mu(d\mathbf{x}, d\mathbf{y}), \quad n = 1, \dots,$$

converges in the weak\* topology of  $L^\infty(\Omega_A, \mathbb{P}_A)$  to an element denoted by the left side of (7.17), whose Walsh series is the right side of (7.17). The  $L^\infty$ -bound in (7.18) is a consequence of the estimate in (7.24).  $\square$

Applying Lemmas 7.4 and 7.5 to (7.14), we obtain

**Proposition 7.6.** *For  $\mu \in \mathcal{F}_2(\mathcal{B}_{l^2}, \mathcal{B}_{l^2})$ ,*

$$\int_{B_{l^2} \times B_{l^2}} \Phi(\mathbf{x}) \Phi(\mathbf{y}) \mu(d\mathbf{x}, d\mathbf{y}) := \sum_{j,k=1}^{\infty} i^{(j+k-2)} \int_{B_{l^2} \times B_{l^2}} \Theta_j(\mathbf{x}) \Theta_k(\mathbf{y}) \mu(d\mathbf{x}, d\mathbf{y}) \quad (7.25)$$

*is an element of  $L^\infty(\Omega_A, \mathbb{P}_A)$ , where the integrals on the right side of (7.25) are given by Lemmas 7.4 and 7.5. Moreover,*

$$\left\| \int_{B_{l^2} \times B_{l^2}} \Phi(\mathbf{x}) \Phi(\mathbf{y}) \mu(d\mathbf{x}, d\mathbf{y}) \right\|_{L^\infty} \leq \left( \frac{2\kappa}{1-\delta} \right)^2 \|\mu\|_{\mathcal{F}_2}. \quad (7.26)$$

**7.3. Integrability of the inner product.** The classical Grothendieck inequality is in effect the statement that the inner product  $\langle \cdot, \cdot \rangle$  in a Hilbert space  $H$  is canonically integrable with respect to all discrete  $\mathcal{F}_2$ -measures on  $B_H \times B_H$ . Namely, by the Grothendieck inequality and basic harmonic analysis, if  $a = (a_{jk}) \in \mathcal{F}_2(\mathbb{N}, \mathbb{N})$ , i.e.,

$$\sup \left\{ \left| \sum_{j,k=1}^N a_{jk} s_j t_k \right| : (s_j, t_k) \in [-1, 1]^2, (j, k) \in [N]^2, N \in \mathbb{N} \right\} := \|a\|_{\mathcal{F}_2} < \infty, \quad (7.27)$$

then for all  $\mathbf{x}_j \in B_H$  and  $\mathbf{y}_k \in B_H$ ,  $j \in \mathbb{N}$ ,  $k \in \mathbb{N}$ ,

$$\begin{aligned} \lim_{N \rightarrow \infty} \sum_{j,k=1}^N a_{jk} \langle \mathbf{x}_j, \mathbf{y}_k \rangle &= \sum_{j=1}^{\infty} \left( \sum_{k=1}^{\infty} a_{jk} \langle \mathbf{x}_j, \mathbf{y}_k \rangle \right) \\ &= \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} a_{jk} \langle \mathbf{x}_j, \mathbf{y}_k \rangle \right) \\ &:= \sum_{j,k=1}^{\infty} a_{jk} \langle \mathbf{x}_j, \mathbf{y}_k \rangle, \end{aligned} \quad (7.28)$$

and

$$\left| \sum_{j,k=1}^{\infty} a_{jk} \langle \mathbf{x}_j, \mathbf{y}_k \rangle \right| \leq \mathcal{K}_G \|a\|_{\mathcal{F}_2}. \quad (7.29)$$

The statements in (7.28) and (7.29) can be equivalently rephrased as follows. Let  $\mathcal{B}_{B_H}$  be the Borel field generated by the weak topology on  $B_H$ . Given  $a = (a_{jk}) \in \mathcal{F}_2(\mathbb{N}, \mathbb{N})$ ,  $\{\mathbf{x}_j\} \subset B_H$ , and  $\{\mathbf{y}_k\} \subset B_H$ , we define  $\mu_a \in \mathcal{F}_2(\mathcal{B}_{B_H}, \mathcal{B}_{B_H})$  by

$$\mu_a = \sum_{j,k} a_{jk} \delta_{\mathbf{x}_j} \otimes \delta_{\mathbf{y}_k}, \quad (7.30)$$

where  $\delta_{\mathbf{x}}$  denotes the usual point mass measure at  $\mathbf{x} \in B_H$ ; i.e.,

$$\mu_a(E, F) = \sum_{\{\mathbf{x}_j \in E, \mathbf{y}_k \in F\}} a_{jk}, \quad (E, F) \in \mathcal{B}_{B_H} \times \mathcal{B}_{B_H},$$

whence

$$\|\mu_a\|_{\mathcal{F}_2(\mathcal{B}_{B_H}, \mathcal{B}_{B_H})} = \|a\|_{\mathcal{F}_2(\mathbb{N}, \mathbb{N})}.$$

Then, the function  $\langle \cdot, \cdot \rangle$  on  $B_H \times B_H$  is integrable with respect to  $\mu_a$ ,

$$\int_{B_H \times B_H} \langle \mathbf{x}, \mathbf{y} \rangle \mu_a(d\mathbf{x}, d\mathbf{y}) := \sum_{j,k=1}^{\infty} a_{jk} \langle \mathbf{x}_j, \mathbf{y}_k \rangle, \quad (7.31)$$

and

$$\left| \int_{B_H \times B_H} \langle \mathbf{x}, \mathbf{y} \rangle \mu_a(d\mathbf{x}, d\mathbf{y}) \right| \leq \mathcal{K}_G \|\mu_a\|_{\mathcal{F}_2}. \quad (7.32)$$



For arbitrary  $\mu \in \mathcal{F}_2(\mathcal{B}_{B_H}, \mathcal{B}_{B_H})$ , if  $H$  is separable and  $A = \{\mathbf{e}_j\}_{j \in \mathbb{N}}$  is an orthonormal basis for it, then Theorem 7.3 implies that the limit

$$\int_{B_H \times B_H} \langle \mathbf{x}, \mathbf{y} \rangle \mu(d\mathbf{x}, d\mathbf{y}) := \lim_{n \rightarrow \infty} \sum_{j=1}^n \int_{B_H \times B_H} \langle \mathbf{x}, \mathbf{e}_j \rangle \langle \mathbf{y}, \mathbf{e}_j \rangle \mu(d\mathbf{x}, d\mathbf{y}) \quad (7.33)$$

exists, and is independent of the basis  $A$ . (E.g., see [12].)

Given a non-separable  $H$ , we fix an orthonormal basis  $A$  for it, and take  $\Phi_A := \Phi$  to be the map supplied by Theorem 3.5. Then, mimicking the argument in (1.6) for  $\mu \in \mathcal{F}_2(\mathcal{B}_{B_H}, \mathcal{B}_{B_H})$ , we formally have

$$\begin{aligned} \int_{B_H \times B_H} \langle \mathbf{x}, \mathbf{y} \rangle \mu(d\mathbf{x}, d\mathbf{y}) &= \int_{B_H \times B_H} \left( \int_{\Omega_A} \Phi_A(\mathbf{x}) \Phi_A(\bar{\mathbf{y}}) d\mathbb{P}_A \right) \mu(d\mathbf{x}, d\mathbf{y}) \\ &:= \int_{\Omega_A} \left( \int_{B_{l^2} \times B_{l^2}} \Phi_A(\mathbf{x}) \Phi_A(\bar{\mathbf{y}}) \mu(d\mathbf{x}, d\mathbf{y}) \right) d\mathbb{P}_A, \end{aligned} \quad (7.34)$$

and

$$\begin{aligned} \left| \int_{B_H \times B_H} \langle \mathbf{x}, \mathbf{y} \rangle \mu(d\mathbf{x}, d\mathbf{y}) \right| &\leq \left\| \int_{B_{l^2} \times B_{l^2}} \Phi_A(\mathbf{x}) \Phi_A(\bar{\mathbf{y}}) \mu(d\mathbf{x}, d\mathbf{y}) \right\|_{L^\infty} \\ &\leq \left( \frac{2\kappa}{1-\delta} \right)^2 \|\mu\|_{\mathcal{F}_2}. \end{aligned} \quad (7.35)$$

**Problem 7.7.** *If  $A$  and  $A'$  are orthonormal bases of  $H$ , and  $\Phi_A$  and  $\Phi_{A'}$  are the respective maps supplied by Theorem 3.5, then does it follow that for every  $\mu \in \mathcal{F}_2(\mathcal{B}_{B_H}, \mathcal{B}_{B_H})$ ,*

$$\begin{aligned} \int_{\Omega_A} \left( \int_{B_{l^2(A)} \times B_{l^2(A)}} \Phi_A(\mathbf{x}) \Phi_A(\bar{\mathbf{y}}) \mu(d\mathbf{x}, d\mathbf{y}) \right) d\mathbb{P}_A \\ = \int_{\Omega_{A'}} \left( \int_{B_{l^2(A')} \times B_{l^2(A')}} \Phi_{A'}(\mathbf{x}) \Phi_{A'}(\bar{\mathbf{y}}) \mu(d\mathbf{x}, d\mathbf{y}) \right) d\mathbb{P}_{A'} ? \end{aligned} \quad (7.36)$$

Indeed, the question whether the inner product in a non-separable  $H$  is integrable with respect to arbitrary  $\mu \in \mathcal{F}_2(\mathcal{B}_{B_H}, \mathcal{B}_{B_H})$  – say, via (7.34) – appears to be open.

### 8. Parseval-like formula for $\langle \mathbf{x}, \mathbf{y} \rangle$ , $\mathbf{x} \in l^p$ , $\mathbf{y} \in l^q$

Next we extend Theorem 3.5 to deal with the dual action between  $l^p$  and  $l^q$  for  $1 \leq p \leq 2 \leq q \leq \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ .

Note at the outset that the Grothendieck inequality, an assertion about the dual action between  $l^2$  and  $l^2$ , cannot be extended verbatim to the general  $l^p - l^q$  setting. Specifically in (1.2), take the scalar array  $(a_{jk})$  to be the Fourier matrix in (2.8), and take  $\mathbf{x}_j$  to be the  $j$ -th basic unit vector  $\mathbf{e}_j$  in  $l^p(\mathbb{N})$ . Then, maximizing over vectors  $\mathbf{y}_k$  in the unit ball of  $l^q(\mathbb{N})$ , we obtain that the left side of (1.2) majorizes

$$\sum_{k=1}^N \left\| \sum_{j=1}^N a_{jk} \mathbf{e}_j \right\|_{l^p} = N^{\frac{1}{p} - \frac{1}{2}}, \quad (8.1)$$

thus verifying that (1.2) fails for  $1 \leq p < 2$ . (Cf. Remark 2.9.i, and also [28], Corollary 1 on p. 289.) In particular, there can be no integral representation of  $\langle \mathbf{x}, \mathbf{y} \rangle$ ,  $\mathbf{x} \in l^p$ ,  $\mathbf{y} \in l^q$ , with uniformly bounded integrands, as in (3.13), for  $1 \leq p < 2 < q \leq \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ .

To derive a Parseval-like formula in the general case, we modify the scheme used to prove Theorem 3.5. Let  $A$  be an infinite set. Define

$$L_{(p)}^\infty(\Omega_A, \mathbb{P}_A) = \{f \in L^\infty(\Omega_A, \mathbb{P}_A) : \hat{f} \in l^p(W_A)\}, \quad 1 \leq p \leq 2, \quad (8.2)$$

and

$$M_{(p)}(\Omega_A) = \{\mu \in M(\Omega_A) : \hat{\mu} \in l^p(W_A)\}, \quad 2 < p \leq \infty, \quad (8.3)$$

with their respective norms

$$\|f\|_{L_{(p)}^\infty} = \max\{\|f\|_{L^\infty}, \|\hat{f}\|_{l^p}\}, \quad f \in L_{(p)}^\infty(\Omega_A, \mathbb{P}_A), \quad 1 \leq p \leq 2, \quad (8.4)$$

and

$$\|\mu\|_{M_{(p)}} = \max\{\|\mu\|_M, \|\hat{\mu}\|_{l^p}\}, \quad \mu \in M_{(p)}(\Omega_A), \quad 2 < p \leq \infty. \quad (8.5)$$

We proceed to construct maps

$$\Phi^{(p)} : l^p(A) \rightarrow L_{(p)}^\infty(\Omega_A, \mathbb{P}_A), \quad 1 \leq p \leq 2, \quad (8.6)$$

and

$$\Phi^{(p)} : l^p(A) \rightarrow M_{(p)}(\Omega_A), \quad 2 < p \leq \infty, \quad (8.7)$$

such that

$$\begin{aligned} \sum_{\alpha \in A} \mathbf{x}(\alpha) \overline{\mathbf{y}(\alpha)} &= \int_{\Omega_A} \Phi^{(p)}(\mathbf{x}) d\Phi^{(q)}(\overline{\mathbf{y}}) \\ &= \sum_{w \in W_A} \widehat{\Phi^{(p)}(\mathbf{x})}(w) \widehat{\Phi^{(q)}(\overline{\mathbf{y}})}(w), \quad \mathbf{x} \in l^p(A), \mathbf{y} \in l^q(A), \end{aligned} \quad (8.8)$$

for all  $1 \leq p \leq 2 \leq q \leq \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , and such that  $\widehat{\Phi^{(p)}}$  is  $(l^p(A) \rightarrow l^p(W_A))$ -continuous for every  $p \in [1, \infty]$ . For  $p = q = 2$ , we let  $d\Phi^{(2)} := \Phi^{(2)}d\mathbb{P}_A$ , and recover Theorem 3.5.

To start, for  $\mathbf{x} \in l^p(A)$  and  $1 \leq p \leq \infty$ , define

$$\sigma_p \mathbf{x} = \begin{cases} \mathbf{x}/\|\mathbf{x}\|_p & \text{if } \mathbf{x} \neq \mathbf{0} \\ \mathbf{0} & \text{if } \mathbf{x} = \mathbf{0}. \end{cases} \quad (8.9)$$

For  $1 < p \leq 2$ , and  $\mathbf{x} \in l_{\mathbb{R}}^p(A)$ ,  $\|\mathbf{x}\|_p = 1$ ,

$$\|Q_A(\mathbf{x})\|_{L^\infty} \leq e^{\frac{1}{2}\|\mathbf{x}\|_2^2} \leq \kappa \quad (8.10)$$

(cf. (4.11)), and

$$\|\widehat{Q_A(\mathbf{x})}|_{R_F^c}\|_p \leq (\sinh(1) - 1)^{\frac{1}{p}} = \delta^{\frac{2}{p}} < 1 \quad (8.11)$$

(cf. (4.13)). Given the setup in §5, we let

$$\mathbf{x}^{(1)}(\tau_0 \alpha) = \mathbf{x}(\alpha), \quad \alpha \in A, \quad (8.12)$$

and define  $\mathbf{x}^{(j)} \in l_{\mathbb{R}}^p(A_j)$ ,  $j = 2, \dots$ , by

$$\mathbf{x}^{(j)}(\tau_{j-1} w) = \|\mathbf{x}^{(j-1)}\|_p \left( Q_{A_{j-1}}(\sigma_p \mathbf{x}^{(j-1)}) \right)^\wedge(w), \quad w \in C_{j-1}. \quad (8.13)$$

By (4.10) and (4.12) in Lemma 4.2, and by (8.10) above, we have for  $j = 1, \dots$ ,

$$\text{spect}(Q_{A_j}(\sigma_p \mathbf{x}^{(j)})) \subset W_{A_j}; \quad (8.14)$$

$$\|Q_{A_j}(\sigma_p \mathbf{x}^{(j)})\|_{L^\infty} \leq \kappa; \quad (8.15)$$

$$\|\mathbf{x}^{(j)}\|_p \left( Q_{A_j}(\sigma_p \mathbf{x}^{(j)}) \right)^\wedge(r_\alpha) = \mathbf{x}^{(j)}(\alpha), \quad \alpha \in A_j. \quad (8.16)$$

Also, by (8.12), (8.13), (8.11), we obtain

$$\|\mathbf{x}^{(j)}\|_p \leq \delta^{\frac{2(j-1)}{p}} \|\mathbf{x}\|_p, \quad j \geq 1. \quad (8.17)$$

Now define

$$\Phi^{(p)}(\mathbf{x}) = \sum_{j=1}^{\infty} i^{j-1} \|\mathbf{x}^{(j)}\|_p Q_{A_j}(\sigma_p \mathbf{x}^{(j)}) \quad (8.18)$$

(cf (5.9)). The series in (8.18) is absolutely convergent in the  $L^\infty$ -norm, and by (8.15) and (8.17),

$$\|\Phi^{(p)}(\mathbf{x})\|_{L^\infty} \leq \left( \frac{\kappa}{1 - \delta^{\frac{2}{p}}} \right) \|\mathbf{x}\|_p. \quad (8.19)$$

By the spectral analysis in (4.7), and by adapting the estimates in (4.8) and (4.9) to the case  $\mathbf{x} \in l^p(A)$ , we deduce

$$\sum_{w \in W_{A_j}} |(Q_{A_j}(\sigma_p \mathbf{x}^{(j)}))^{\wedge}(w)|^p \leq 1 + \delta^2, \quad (8.20)$$

and therefore,

$$\|\widehat{\Phi^{(p)}(\mathbf{x})}\|_p \leq \left( \frac{1 + \delta^2}{1 - \delta^2} \right)^{\frac{1}{p}} \|\mathbf{x}\|_p. \quad (8.21)$$

By (8.19) and (8.21), we obtain

$$\begin{aligned} \|\Phi^{(p)}(\mathbf{x})\|_{L_{(p)}^\infty} &\leq \max \left\{ \frac{\kappa}{1 - \delta^{\frac{2}{p}}}, \left( \frac{1 + \delta^2}{1 - \delta^2} \right)^{\frac{1}{p}} \right\} \|\mathbf{x}\|_p. \\ &\leq \left( \frac{\kappa}{1 - \delta^{\frac{2}{p}}} \right) \|\mathbf{x}\|_p. \end{aligned} \quad (8.22)$$

Next we let  $2 < p < \infty$ , and  $\mathbf{x} \in l_{\mathbb{R}}^p(A)$ ,  $\|\mathbf{x}\|_p = 1$ . For  $F \subset A$ , define

$$\begin{aligned} P_F(\mathbf{x}) &= \frac{\Re_F(\mathbf{x}) - \Re_F(-\mathbf{x})}{2} \\ &\sim \sum_{k=0}^{\infty} \left( \sum_{\substack{w \in W_{F, 2k+1} \\ w = r_{\alpha_1} \cdots r_{\alpha_{2k+1}}}} \mathbf{x}(\alpha_1) \cdots \mathbf{x}(\alpha_{2k+1}) r_{\alpha_1} \cdots r_{\alpha_{2k+1}} \right), \end{aligned} \quad (8.23)$$

where  $\Re_F(\mathbf{x})$  is the Riesz product in §4.4.i. Then,  $P_F(\mathbf{x}) \in B_{M(\Omega_A)}$ , and otherwise, the estimates involving  $\widehat{P_F(\mathbf{x})}$  are identical to those involving  $\widehat{Q_F(\mathbf{x})}$ . We define  $\mathbf{x}^{(1)} \in l^p(A_1)$  by (8.12), and  $\mathbf{x}^{(j)} \in l_{\mathbb{R}}^p(A_j)$ ,  $j = 2, \dots$ , by

$$\mathbf{x}^{(j)}(\tau_{j-1}w) = \|\mathbf{x}^{(j-1)}\|_p (P_{A_{j-1}}(\sigma_p \mathbf{x}^{(j-1)}))^{\wedge}(w), \quad w \in C_{j-1}, \quad (8.24)$$

and then define

$$\Phi^{(p)}(\mathbf{x}) = \sum_{j=1}^{\infty} i^{j-1} \|\mathbf{x}^{(j)}\|_p P_{A_j}(\sigma_p \mathbf{x}^{(j)}), \quad 2 < p < \infty. \quad (8.25)$$

We obtain  $\Phi^{(p)}(\mathbf{x}) \in M_{(p)}(\Omega_A)$ , and

$$\|\Phi^{(p)}(\mathbf{x})\|_{M_{(p)}} \leq \left( \frac{1 + \delta^2}{1 - \delta^2} \right)^{\frac{1}{p}} \|\mathbf{x}\|_p. \quad (8.26)$$

For  $1 < p < \infty$  and  $\mathbf{x} \in l^p(A)$ , define (cf. (5.10))

$$\Phi^{(p)}(\mathbf{x}) := \Phi^{(p)}(\mathbf{u}) + i\Phi^{(p)}(\mathbf{v}), \quad \mathbf{x} = \mathbf{u} + i\mathbf{v}, \quad \mathbf{u} \in l_{\mathbb{R}}^p(A), \quad \mathbf{v} \in l_{\mathbb{R}}^p(A). \quad (8.27)$$

The proof of the integral representation in (8.8) is essentially the same as the proof of (3.13) in §5.3. The proof of the homogeneity property,

$$\Phi^{(p)}(c\mathbf{x}) = c\Phi^{(p)}(\mathbf{x}), \quad c \in \mathbb{R}, \quad \mathbf{x} \in l^p(A), \quad (8.28)$$

is also the same as the proof of (3.12).

To prove that  $\widehat{\Phi^{(p)}}$  is  $(l^p(A) \rightarrow l^p(W_A))$ -continuous for  $1 < p < \infty$ , we use (8.28) to verify continuity at  $\mathbf{x} = \mathbf{0}$ , and otherwise modify the proof of Lemma 5.1 to show that if  $\mathbf{x} \in B_{l_{\mathbb{R}}^p(A)}$ ,  $\mathbf{y} \in B_{l_{\mathbb{R}}^p(A)}$ ,  $\mathbf{x} \neq \mathbf{0}$ , and  $\mathbf{y} \neq \mathbf{0}$ , then

$$3^{\frac{1}{p}} \|\mathbf{x} - \mathbf{y}\|_p \geq \begin{cases} \|\widehat{Q_A(\mathbf{x})} - \widehat{Q_A(\mathbf{y})}\|_p & 1 < p \leq 2 \\ \|\widehat{P_A(\mathbf{x})} - \widehat{P_A(\mathbf{y})}\|_p & 2 < p < \infty. \end{cases} \quad (8.29)$$

For the extremal instances  $p = 1$  and  $p = \infty$ , define

$$\Phi^{(1)} : l^1(A) \rightarrow L_{(1)}^\infty(\Omega_A, \mathbb{P}_A)$$

and

$$\Phi^{(\infty)} : l^{(\infty)}(A) \rightarrow M(\Omega_A),$$

by

$$\Phi^{(1)}(\mathbf{x}) = \sum_{\alpha \in A} \mathbf{x}(\alpha) r_\alpha, \quad \mathbf{x} \in l^1(A), \quad (8.30)$$

and

$$\Phi^{(\infty)}(\mathbf{x}) = \|\mathbf{u}\|_\infty P_A(\sigma_\infty \mathbf{u}) + i\|\mathbf{v}\|_\infty P_A(\sigma_\infty \mathbf{v}), \quad \mathbf{x} = \mathbf{u} + i\mathbf{v}, \quad \mathbf{u} \in l_{\mathbb{R}}^\infty(A), \quad \mathbf{v} \in l_{\mathbb{R}}^\infty(A).$$

The integral representation in (8.8) for  $p = 1$  and  $q = \infty$  is the usual Parseval formula. The  $(l^1(A) \rightarrow l^1(W_A))$ -continuity of  $\Phi^{(1)}$  is obvious, and the  $(l^\infty(A) \rightarrow l^\infty(W_A))$ -continuity of  $\Phi^{(\infty)}$  follows from the spectral analysis of  $P_A$ . We summarize.

**Theorem 8.1.** *Let  $A$  be an infinite set. There exist one-one maps*

$$\Phi^{(p)} : l^p(A) \rightarrow \begin{cases} L_{(p)}^\infty(\Omega_A, \mathbb{P}_A), & 1 \leq p \leq 2, \\ M_{(p)}(\Omega_A), & 2 < p \leq \infty, \end{cases} \quad (8.31)$$

*with the following properties:*

$$\begin{aligned} \|\Phi^{(p)}(\mathbf{x})\|_{L_{(p)}^\infty} &\leq \left(\frac{\kappa}{1-\delta^{\frac{2}{p}}}\right)\|\mathbf{x}\|_p, \quad \mathbf{x} \in l^p(A), \quad 1 \leq p \leq 2 \\ \|\Phi^{(p)}(\mathbf{x})\|_{M_{(p)}} &\leq \left(\frac{1+\delta^2}{1-\delta^2}\right)^{\frac{1}{p}}\|\mathbf{x}\|_p, \quad \mathbf{x} \in l^p(A), \quad 2 < p \leq \infty; \end{aligned} \quad (8.32)$$

$$\Phi^{(p)}(c\mathbf{x}) = c\Phi^{(p)}(\mathbf{x}), \quad c \in \mathbb{R}, \quad \mathbf{x} \in l^p(A); \quad (8.33)$$

$$\begin{aligned} \sum_{\alpha \in A} \mathbf{x}(\alpha) \overline{\mathbf{y}(\alpha)} &= \int_{\Omega_A} \Phi^{(p)}(\mathbf{x}) d\Phi^{(q)}(\overline{\mathbf{y}}) = \sum_{w \in W_A} \widehat{\Phi^{(p)}(\mathbf{x})}(w) \widehat{\Phi^{(q)}(\overline{\mathbf{y}})}(w), \\ \mathbf{x} &\in l^p(A), \quad \mathbf{y} \in l^q(A), \quad \frac{1}{p} + \frac{1}{q} = 1, \quad 1 \leq p \leq 2 \leq q \leq \infty \end{aligned} \quad (8.34)$$

$(d\Phi^{(2)} := \Phi^{(2)}d\mathbb{P}_A)$ , and the maps  $\widehat{\Phi^{(p)}}$  are  $(l^p(W) \rightarrow l^p(W_A))$ -continuous for all  $1 \leq p \leq \infty$ .

## 9. Grothendieck-like theorems in dimensions $> 2$ ?

The general focus so far has been on scalar-valued functions of two variables and their representations by functions of one variable. Next we consider functions of  $n$  variables,  $n \geq 2$ , and examine, analogously, the feasibility of representing them by functions that depend on  $k$  variables,  $0 < k < n$ .

**9.1. The multidimensional version of Question 2.3.** We begin with a natural extension of Question 2.3 for arbitrary  $n \geq 2$  and  $k = 1$ . Let  $X_1, \dots, X_n$  be sets, and denote

$$\mathbf{X}^{[n]} := X_1 \times \dots \times X_n.$$

**Question 9.1.** *Given  $f \in l^\infty(\mathbf{X}^{[n]})$ , can we find a probability space  $(\Omega, \mu)$ , and functions  $g_\omega^{(i)}$  indexed by  $\omega \in \Omega$  and defined on  $X_i$ ,  $i \in [n]$ , such that for all  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{X}^{[n]}$  and  $i \in [n]$ ,*

$$\omega \mapsto g_\omega^{(i)}(x_i) \quad \text{and} \quad \omega \mapsto \|g_\omega^{(i)}\|_\infty$$

*are  $\mu$ -measurable on  $\Omega$ , and*

$$f(\mathbf{x}) = \int_{\Omega} g_\omega^{(1)}(x_1) \cdots g_\omega^{(n)}(x_n) \mu(d\omega) \quad (9.1)$$

*under the constraint*

$$\sup_{\mathbf{x} \in \mathbf{X}^{[n]}} \|g_\omega^{(1)}(x_1)\|_{L^\infty(\mu)} \cdots \|g_\omega^{(n)}(x_n)\|_{L^\infty(\mu)} < \infty? \quad (9.2)$$

(As before, we refer to  $\Omega$  as an indexing space, to  $\mu$  as an indexing measure, and to  $g_\omega^{(i)}$  as representing functions.)

Matters are formalized as usual. Given  $f \in l^\infty(\mathbf{X}^{[n]})$ , let  $\|f\|_{\tilde{\mathcal{V}}_n}$  be the infimum of the left side of (9.2) taken over all probability spaces  $(\Omega, \mu)$ , and all families of functions on  $X_i$  indexed by  $(\Omega, \mu)$  that represent  $f$  by (9.1). We obtain the Banach algebra

$$\tilde{\mathcal{V}}_n(\mathbf{X}^{[n]}) = \{f \in l^\infty(\mathbf{X}^{[n]}) : \|f\|_{\tilde{\mathcal{V}}_n} < \infty\},$$

wherein algebra multiplication is given by point-wise multiplication on  $\mathbf{X}^{[n]}$ . It follows from the case  $n = 2$  that if at least two of the  $X_i$  are infinite, then  $\tilde{\mathcal{V}}_n(\mathbf{X}^{[n]}) \subsetneq l^\infty(\mathbf{X}^{[n]})$ .

Proposition 2.4 has a straightforward extension. Take the compact Abelian group

$$\Omega_{\mathbf{X}^{[n]}} := \Omega_{X_1} \times \cdots \times \Omega_{X_n},$$

and the spectral set ( $n$ -fold Cartesian product of Rademacher systems)

$$\begin{aligned} \mathbf{R}_{\mathbf{X}^{[n]}} &= R_{X_1} \times \cdots \times R_{X_n} \\ &= \{r_{x_1} \otimes \cdots \otimes r_{x_n} : (x_1, \dots, x_n) \in \mathbf{X}^{[n]}\} \subset \hat{\Omega}_{\mathbf{X}^{[n]}}. \end{aligned}$$

Then, by identifying

$$\begin{aligned} B(\mathbf{R}_{\mathbf{X}^{[n]}}) &:= \widehat{M}(\Omega_{\mathbf{X}^{[n]}}) / \{\hat{\lambda} \in \widehat{M}(\Omega_{\mathbf{X}^{[n]}}) : \hat{\lambda} = 0 \text{ on } \mathbf{R}_{\mathbf{X}^{[n]}}\} \\ &= \{\text{restrictions of Walsh transforms to } \mathbf{R}_{\mathbf{X}^{[n]}}\} \end{aligned} \tag{9.3}$$

as the dual space of

$$C_{\mathbf{R}_{\mathbf{X}^{[n]}}}(\Omega_{\mathbf{X}^{[n]}}) := \{\text{continuous functions on } \Omega_{\mathbf{X}^{[n]}} \text{ with spectrum in } \mathbf{R}_{\mathbf{X}^{[n]}}\}, \tag{9.4}$$

we obtain (omitting the proof)

**Proposition 9.2.**

$$\tilde{\mathcal{V}}_n(\mathbf{X}^{[n]}) = B(\mathbf{R}_{\mathbf{X}^{[n]}}).$$

**9.2. A multidimensional version of Question 2.6.** Next we consider integral representations of  $f \in l^\infty(\mathbf{X}^{[n]})$  with constraints weaker than (9.2), and ask whether the feasibility of such representations implies the feasibility of integral representations under the stronger constraint in (9.2). That is, are there Grothendieck-like theorems in dimensions  $> 2$ ? (Cf. discussion preceding Theorem 2.7, and also Remark 2.9.i.)

Attempting first the obvious, we extend Question 2.6 by replacing in it the role of Cauchy-Schwarz with that of a multi-linear Hölder inequality.

**Question 9.3.** *Let  $f \in l^\infty(\mathbf{X}^{[n]})$ , and let  $\mathbf{p} = (p_1, \dots, p_n)$  be a conjugate vector, i.e.,  $\mathbf{p} = (p_1, \dots, p_n) \in [1, \infty]^n$  such that*

$$\frac{1}{p_1} + \dots + \frac{1}{p_n} = 1.$$

*Can we find a probability space  $(\Omega, \mu)$ , and functions  $g_\omega^{(i)}$  indexed by  $\omega \in \Omega$  and defined on  $X_i$ ,  $i \in [n]$ , such that for  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{X}^{[n]}$  and  $i \in [n]$ ,*

$$\omega \mapsto g_\omega^{(i)}(x_i)$$

*determine elements in  $L^{p_i}(\Omega, \mu)$ , and*

$$f(\mathbf{x}) = \int_{\Omega} g_\omega^{(1)}(x_1) \cdots g_\omega^{(n)}(x_n) \mu(d\omega) \quad (9.5)$$

*subject to*

$$\sup_{\mathbf{x} \in \mathbf{X}^{[n]}} \|g_\omega^{(1)}(x_1)\|_{L^{p_1}(\mu)} \cdots \|g_\omega^{(n)}(x_n)\|_{L^{p_n}(\mu)} < \infty? \quad (9.6)$$

For  $f \in l^\infty(\mathbf{X}^{[n]})$ , define  $\|f\|_{\mathcal{G}_{\mathbf{p}}}$  as the infimum of the left side of (9.6) over all integral representations of  $f$  by (9.5). We obtain the Banach algebra

$$\mathcal{G}_{\mathbf{p}}(\mathbf{X}^{[n]}) = \{f \in l^\infty(\mathbf{X}^{[n]}) : \|f\|_{\mathcal{G}_{\mathbf{p}}} < \infty\}, \quad (9.7)$$

and note the inclusion

$$\tilde{\mathcal{V}}_n(\mathbf{X}^{[n]}) \subset \mathcal{G}_{\mathbf{p}}(\mathbf{X}^{[n]}), \quad n \geq 2. \quad (9.8)$$

If  $n = 2$  and  $\mathbf{p} = (2, 2)$ , then also the reverse inclusion holds; this is the gist of Theorem 2.7. But otherwise, if  $X_1$  and  $X_2$  are infinite and  $\mathbf{p} \neq (2, 2)$ , then (9.8) is a proper inclusion; see Remark 2.9.i, and also the start of §8.



For  $n > 2$  and infinite  $X_i$ ,  $i \in [n]$ , the inclusion in (9.8) is *always* proper. We verify this in the case  $n = 3$ , which is typical.

Let  $\mathbf{p} = (p_1, p_2, p_3) \in [1, \infty]^3$  be a conjugate vector, and suppose  $p_3 < \infty$ . For all integers  $N > 1$  and  $(\omega_1, \omega_2, \omega_3) \in \{-1, 1\}^N \times \{-1, 1\}^N \times \{-1, 1\}^N$ ,

$$\left| \frac{1}{N^2} \sum_{j,k,l=1}^N e^{2\pi i \frac{(j+k)l}{N}} r_j(\omega_1) r_k(\omega_2) r_l(\omega_3) \right| \leq 1. \quad (9.9)$$

(Cf. (2.8).) Take  $X_i = [N]$  for  $i = 1, 2, 3$ , and  $\Omega = [N]$  with the uniform probability measure  $\mu$  on it, and consider representing functions

$$g_\omega^{(1)}(j) = e^{\frac{-2\pi i j \omega}{N}}, \quad g_\omega^{(2)}(k) = e^{\frac{-2\pi i k \omega}{N}}, \quad \omega \in \Omega, \quad j \in X_1, \quad k \in X_2,$$

and

$$g_\omega^{(3)}(l) = \begin{cases} 0 & \text{if } \omega \neq l \\ N^{\frac{1}{p_3}} & \text{if } \omega = l, \end{cases} \quad \omega \in \Omega, \quad l \in X_3.$$

Then,

$$f(j, k, l) := \int_{\Omega} g_\omega^{(1)}(j) g_\omega^{(2)}(k) g_\omega^{(3)}(l) \mu(d\omega) = N^{\frac{1}{p_3}-1} e^{-2\pi i \frac{(j+k)l}{N}}, \quad (j, k, l) \in \mathbf{X}^{[3]},$$

and

$$\sup_{(j,k,l) \in \mathbf{X}^{[3]}} \|g_\omega^{(1)}(j)\|_{L^{p_1}(\mu)} \|g_\omega^{(2)}(k)\|_{L^{p_2}(\mu)} \|g_\omega^{(3)}(l)\|_{L^{p_3}(\mu)} = 1.$$

By duality and (9.9), we obtain

$$\|f\|_{\tilde{\mathcal{V}}_3} \geq \left| \frac{1}{N^2} \sum_{j,k,l=1}^N f(j, k, l) e^{-2\pi i \frac{(j+k)l}{N}} \right| = N^{\frac{1}{p_3}} \xrightarrow{N \rightarrow \infty} \infty,$$

and conclude that if  $X_1$ ,  $X_2$ , and  $X_3$  are infinite sets, then we have proper inclusions

$$\tilde{\mathcal{V}}_3(\mathbf{X}^{[3]}) \subsetneq \mathcal{G}_{(p_1, p_2, p_3)}(\mathbf{X}^{[3]}) \subsetneq l^\infty(\mathbf{X}^{[3]}),$$

and a question remains: what are multidimensional analogues of the "two-dimensional" inclusion

$$\mathcal{G}_{(2,2)}(\mathbf{X}^{[2]}) \subset \tilde{\mathcal{V}}_2(\mathbf{X}^{[2]})?$$

## 10. Fractional Cartesian products and multilinear functionals on a Hilbert space

We return to a Hilbert space setting as staging grounds for Grothendieck-like inequalities.

**10.1. Projective boundedness and projective continuity.** To start, we note that the (two-dimensional) Grothendieck theorem is an assertion about the feasibility of integral representations of bilinear functionals on a Hilbert space: if  $\eta$  is a bounded bilinear functional on a Hilbert space  $H$ , then there exist bounded mappings  $\phi_1$  and  $\phi_2$  from  $B_H$  into  $L^\infty(\Omega, \mu)$ , for some probability space  $(\Omega, \mu)$ , such that

$$\eta(\mathbf{x}, \mathbf{y}) = \int_{\Omega} \phi_1(\mathbf{x}) \phi_2(\mathbf{y}) d\mu, \quad (\mathbf{x}, \mathbf{y}) \in (B_H)^2.$$

(Cf. (1.5) in Proposition 1.1.) Theorem 3.5 provides an upgrade, asserting that  $\phi_1$  and  $\phi_2$  can be chosen to be continuous with respect to the norm topologies on  $H$  and  $L^2(\Omega, \mu)$ .

**Definition 10.1.** Let  $\eta$  be a bounded  $n$ -linear functional on a Hilbert space  $H$ , i.e.,

$$\eta : \underbrace{H \times \cdots \times H}_n \longrightarrow \mathbb{C}$$

is linear in each coordinate and

$$\|\eta\|_\infty := \sup\{|\eta(\mathbf{x}_1, \dots, \mathbf{x}_n)| : (\mathbf{x}_1, \dots, \mathbf{x}_n) \in (B_H)^n\} < \infty.$$

**i.**  $\eta$  is projectively bounded if there exist a probability space  $(\Omega, \mu)$  and bounded mappings  $\phi_1, \dots, \phi_n$  from  $B_H$  into  $L^\infty(\Omega, \mu)$ , such that

$$\eta(\mathbf{x}_1, \dots, \mathbf{x}_n) = \int_{\Omega} \phi_1(\mathbf{x}_1) \cdots \phi_n(\mathbf{x}_n) d\mu, \quad (\mathbf{x}_1, \dots, \mathbf{x}_n) \in (B_H)^n. \quad (10.1)$$

Equivalently,  $\eta$  is projectively bounded if

$$\|\eta\|_{\tilde{V}_n(B_H \times \cdots \times B_H)} < \infty. \quad (10.2)$$

**ii.**  $\eta$  is projectively continuous if there exist a probability space  $(\Omega, \mu)$  and bounded mappings  $\phi_1, \dots, \phi_n$  from  $B_H$  into  $L^\infty(\Omega, \mu)$  that satisfy (10.1), and are also  $(H \rightarrow L^2(\Omega, \mu))$ -continuous. Equivalently,  $\eta$  is projectively continuous if

$$\|\eta\|_{\tilde{V}_n(B_H \times \cdots \times B_H)} < \infty, \quad (10.3)$$

where the  $\|\cdot\|_{\tilde{V}_n(\mathbf{X}^{[n]})}$ -norm is the  $n$ -dimensional extension of the  $\|\cdot\|_{\tilde{V}_2(X_1 \times X_2)}$ -norm in Definition 3.2.

(The notion of projective boundedness is equivalent, through duality, to the notion in [5, Definition 1.1].) We have

$$\|\eta\|_{\tilde{V}_n(B_H \times \cdots \times B_H)} \leq \|\eta\|_{\tilde{V}_n(B_H \times \cdots \times B_H)}. \quad (10.4)$$

Projective continuity implies projective boundedness, but I do not know whether the converse holds. (See §12.4.)

In the case  $n = 1$ , every bounded linear functional on a Hilbert space  $H$  is projectively continuous. Obviously an instance of the "two-dimensional" Theorem 3.5, the "one-dimensional" precursor is not entirely obvious by itself. To wit, the ostensibly weaker assertion that every bounded linear functional on a Hilbert space is projectively bounded is equivalent to the (so-called) *little Grothendieck Theorem* [31, Theorem 5.1]. (See Remark 10.4 below.)

**Proposition 10.2.** *If  $\mathbf{y} \in B_H$ , and*

$$\eta_{\mathbf{y}}(\mathbf{x}) = \langle \mathbf{x}, \mathbf{y} \rangle, \quad \mathbf{x} \in B_H, \quad (10.5)$$

*where  $\langle \cdot, \cdot \rangle$  is the inner product in  $H$ , then*

$$\|\eta_{\mathbf{y}}\|_{\tilde{V}_1(B_H)} \leq 2e^{\frac{1}{2}}. \quad (10.6)$$

*Proof.* For simplicity (and with no loss of generality) assume  $H = l_{\mathbb{R}}^2(A)$ . Define

$$g_{\mathbf{y}} = \sum_{\alpha \in A} \mathbf{y}(\alpha) r_{\alpha}. \quad (10.7)$$

Then,  $g_{\mathbf{y}} \in L_{R_A}^2(\Omega_A, \mathbb{P}_A)$ , and

$$\|\mathbf{y}\|_2 = \|g_{\mathbf{y}}\|_{L^2} \leq 1. \quad (10.8)$$

By Parseval's formula and the spectral analysis of Riesz products (as per §4), we have

$$\eta_{\mathbf{y}}(\mathbf{x}) = \sum_{\alpha \in A} \mathbf{x}(\alpha) \mathbf{y}(\alpha) = \int_{\Omega_A} Q_A(\mathbf{x}) g_{\mathbf{y}} d\mathbb{P}_A, \quad \mathbf{x} \in B_{l^2}, \quad (10.9)$$

which, by (4.11) and Lemma 5.1, implies

$$\|\eta_{\mathbf{y}}\|_{\tilde{V}_1(B_H)} \leq e^{\frac{1}{2}}. \quad (10.10)$$

□

In the case  $n = 2$ , Theorem 3.5 implies that every bounded *bilinear* functional on a Hilbert space is projectively continuous.

In the case  $n = 3$ , there exist bounded trilinear functionals that are *not* projectively bounded [42], and *a fortiori* not projectively continuous. (See Remark 11.7.)

**Problem 10.3.** *For an infinite-dimensional Hilbert space  $H$ , and for  $n > 2$ , which are the projectively bounded, and which are the projectively continuous  $n$ -linear functionals on  $H$ ?*

We may as well take  $H$  to be  $l^2(A)$ , where  $A$  is an infinite indexing set. Then, given a bounded  $n$ -linear functional  $\eta$  on  $l^2(A)$ , we write (first formally)

$$\eta(\mathbf{x}_1, \dots, \mathbf{x}_n) = \sum_{(\alpha_1, \dots, \alpha_n) \in A^n} \theta_{A, \eta}(\alpha_1, \dots, \alpha_n) \mathbf{x}_1(\alpha_1) \cdots \mathbf{x}_n(\alpha_n), \quad (10.11)$$

$$\mathbf{x}_1 \in l^2(A), \dots, \mathbf{x}_n \in l^2(A),$$

where

$$\theta_{A, \eta}(\alpha_1, \dots, \alpha_n) := \eta(\mathbf{e}_{\alpha_1}, \dots, \mathbf{e}_{\alpha_n}), \quad (\alpha_1, \dots, \alpha_n) \in A^n, \quad (10.12)$$

and  $\{\mathbf{e}_\alpha : \alpha \in A\}$  is the standard basis of  $l^2(A)$ ; cf. (1.36). In general, the sum on the right side of (10.11) converges *conditionally*, and is computed iteratively, coordinate-by-coordinate. That is,

$$\eta(\mathbf{x}_1, \dots, \mathbf{x}_n) = \sum_{\alpha_1 \in A} \left( \cdots \left( \sum_{\alpha_n \in A} \theta_{A, \eta}(\alpha_1, \dots, \alpha_n) \mathbf{x}_1(\alpha_1) \cdots \mathbf{x}_n(\alpha_n) \right) \cdots \right) \quad (10.13)$$

$$\mathbf{x}_1 \in l^2(A), \dots, \mathbf{x}_n \in l^2(A),$$

where sums over the respective coordinates can be performed iteratively in any order. In this work we consider bounded  $n$ -linear functionals for which the right side of (10.11) is absolutely summable, focusing specifically on kernels  $\theta_{A, \eta}$  that are supported by *fractional Cartesian products* of a certain type.

**Remark 10.4.** The little Grothendieck Theorem is equivalent to the assertion that there exists a constant  $K > 1$  such that for every finite scalar array  $(a_{jk})$ ,

$$\sup \left\{ \left| \sum_{j,k} a_{jk} \langle \mathbf{x}_j, \mathbf{y}_k \rangle \right| : \mathbf{x}_j \in l^2, \mathbf{y}_k \in l^2, \|\mathbf{x}_j\|_2 \leq 1, \sum_k \|\mathbf{y}_k\|_2^2 \leq 1 \right\} \quad (10.14)$$

$$\leq K \sup \left\{ \left| \sum_{j,k} a_{jk} s_j t_k \right| : |s_j| \leq 1, \sum_k |t_k|^2 \leq 1 \right\}.$$

(Cf. [31, Theorem 5.2].) Notably, whereas the inequality in (10.14) is a quick consequence of the inequality in (1.2), it is implied also – though not as quickly – by

$$\sup \{ \|\eta_{\mathbf{y}}\|_{\tilde{V}_1(B_H)} : \mathbf{y} \in B_H \} := k_G < \infty, \quad (10.15)$$

where  $\eta_{\mathbf{y}}$  is defined in (10.5), and  $H$  is an infinite-dimensional Hilbert space. E.g., (10.14) is implied by the integral representation in (10.9).

Conversely, the statement that (10.14) holds for every finite scalar array implies that every bounded linear functional on a Hilbert space is projectively bounded. Indeed,  $k_G$  in (10.15) is equal to the "best"  $K$  in (10.14). In particular, by identifying a separable Hilbert space with the  $L^2$ -closure of the linear span of independent standard normal random variables, we obtain (essentially by re-writing the proof of Proposition 10.2)  $k_G := k_G^{\mathbb{C}} = \sqrt{4/\pi}$  if scalars are complex numbers, and  $k_G := k_G^{\mathbb{R}} = \sqrt{\pi/2}$  in the case of the reals. (Cf. [31, Lemma 5.3].)

With the requirement that the action of  $\eta_{\mathbf{y}}$  be represented by integrals with uniformly bounded *norm-continuous* integrands, as in (10.9), we have

$$k_G \leq \sup\{\|\eta_{\mathbf{y}}\|_{\tilde{V}_1(B_H)} : \mathbf{y} \in B_H, \text{ Hilbert space } H\} := k_{CG} \leq 2e^{\frac{1}{2}}. \quad (10.16)$$

I do not know which of the inequalities above are strict; cf. (10.4) and Problem 12.5.

**10.2. Fractional Cartesian products.** Let  $m$  be a positive integer. A *covering sequence* of  $[m]$  is a set-valued sequence  $\mathcal{U} = (S_1, \dots, S_n)$ , such that  $\emptyset \neq S_i \subset [m]$ , and

$$\bigcup_{i=1}^n S_i = [m].$$

Given  $S \subset [m]$  and sets  $X_1, \dots, X_m$ , we consider the projection

$$\pi_S : \bigtimes_{j=1}^m X_j \rightarrow \bigtimes_{j \in S} X_j := \mathbf{X}^S, \quad (10.17)$$

defined by

$$\pi_S(\mathbf{x}) = (x_j : j \in S), \quad \mathbf{x} = (x_1, \dots, x_m) \in \bigtimes_{j=1}^m X_j. \quad (10.18)$$

Then, given a covering sequence  $\mathcal{U} = (S_1, \dots, S_n)$  of  $[m]$ , we define

$$\begin{aligned} \mathbf{X}^{\mathcal{U}} &:= \{(\pi_{S_1}(\mathbf{x}), \dots, \pi_{S_n}(\mathbf{x})) : \mathbf{x} \in \mathbf{X}^{[m]}\} \\ &\subset \mathbf{X}^{S_1} \times \dots \times \mathbf{X}^{S_n} := \mathbf{X}^{[\mathcal{U}]}. \end{aligned} \quad (10.19)$$

We refer to  $\mathbf{X}^{\mathcal{U}}$  as a *fractional Cartesian product* (based on  $\mathcal{U}$ ), and to  $\mathbf{X}^{[\mathcal{U}]}$  as its *ambient product*.

**Example 10.5** (*3/2-product*). If  $\mathcal{U} = (S_1, \dots, S_n)$  covers  $[m]$ , and the  $S_i$  are pairwise disjoint, then  $\mathbf{X}^{\mathcal{U}} = \mathbf{X}^{[\mathcal{U}]}$ . For our purposes here, we will consider sequences  $\mathcal{U}$  with the

property that every  $i \in [m]$  appears in at least two elements of  $\mathcal{U}$ . The simplest nontrivial example is

$$\mathcal{U} = (\{1, 2\}, \{2, 3\}, \{1, 3\}). \quad (10.20)$$

In this instance, given sets  $X_1, X_2, X_3$ , we have

$$\mathbf{X}^{\mathcal{U}} = \{((x_1, x_2), (x_2, x_3), (x_1, x_3)) : (x_1, x_2, x_3) \in X_1 \times X_2 \times X_3\}, \quad (10.21)$$

and

$$\mathbf{X}^{[\mathcal{U}]} = (X_1 \times X_2) \times (X_2 \times X_3) \times (X_1 \times X_3). \quad (10.22)$$

We view  $\mathbf{X}^{[\mathcal{U}]}$  as a three-fold Cartesian product, and  $\mathbf{X}^{\mathcal{U}}$  as a "3/2-fold" Cartesian product. (The appearance of the fraction 3/2 is explained in Remarks 10.8.i and 10.10.i.) We will use the "3/2-fold" Cartesian product as a running example, to illustrate the workings in general definitions and arguments.

Fractional Cartesian products provide a natural framework for the study of representing functions of  $m$  variables,  $m > 2$ , by functions of  $k$  variables,  $1 < k < m$ .

**Question 10.6** (cf. Question 9.3). *Let  $\mathcal{U} = (S_1, \dots, S_n)$  be a covering sequence of  $[m]$ . Given  $f \in l^\infty(\mathbf{X}^{[m]})$ , can we find a probability space  $(\Omega, \mu)$ , and representing functions  $g_\omega^{(i)}$ , indexed by  $\omega \in \Omega$  and defined on  $\mathbf{X}^{S_i}$  for  $i \in [n]$ , such that for all  $\mathbf{x} \in \mathbf{X}^{[m]}$*

$$\omega \mapsto (g_\omega^{(i)} \circ \pi_{S_i})(\mathbf{x}) \quad \text{and} \quad \omega \mapsto \|g_\omega^{(i)}\|_\infty$$

*determine  $\mu$ -measurable functions on  $\Omega$ , and*

$$f(\mathbf{x}) = \int_{\Omega} (g_\omega^{(1)} \circ \pi_{S_1})(\mathbf{x}) \cdots (g_\omega^{(n)} \circ \pi_{S_n})(\mathbf{x}) \mu(d\omega) \quad (10.23)$$

*under the constraint*

$$\int_{\Omega} \|g_\omega^{(1)}\|_\infty \cdots \|g_\omega^{(n)}\|_\infty \mu(d\omega) < \infty? \quad (10.24)$$

We let  $\|f\|_{\tilde{\mathcal{V}}_{\mathcal{U}}}$  be the infimum of the left side of (10.24) taken over all representations of  $f$  by (10.23), and define

$$\tilde{\mathcal{V}}_{\mathcal{U}}(\mathbf{X}^{[m]}) := \{f \in l^\infty(\mathbf{X}^{[m]}) : \|f\|_{\tilde{\mathcal{V}}_{\mathcal{U}}} < \infty\}. \quad (10.25)$$

Extending Proposition 9.2, we identify  $\tilde{\mathcal{V}}_{\mathcal{U}}(\mathbf{X}^{[m]})$  with an algebra of restrictions of Walsh transforms. Specifically, we take the compact Abelian group

$$\Omega_{\mathbf{X}^{[\mathcal{U}]}} := \Omega_{\mathbf{X}^{S_1}} \times \cdots \times \Omega_{\mathbf{X}^{S_n}}, \quad (10.26)$$

and the spectral set

$$(\mathbf{R}_{\mathbf{X}})^{\mathcal{U}} := \{r_{\pi_{S_1}(\mathbf{x})} \otimes \cdots \otimes r_{\pi_{S_l}(\mathbf{x})} : \mathbf{x} \in \mathbf{X}^{[m]}\}, \quad (10.27)$$

which is a fractional Cartesian product, whose ambient product is

$$(\mathbf{R}_{\mathbf{X}})^{[\mathcal{U}]} = R_{\mathbf{X}^{S_1}} \times \cdots \times R_{\mathbf{X}^{S_n}}. \quad (10.28)$$

(See (10.19).) In the case of the 3/2-product (Example 10.5 above), we have

$$\mathbf{\Omega}_{\mathbf{X}^{[\mathcal{U}]}} = \{-1, 1\}^{X_1 \times X_2} \times \{-1, 1\}^{X_2 \times X_3} \times \{-1, 1\}^{X_1 \times X_3}, \quad (10.29)$$

and

$$(\mathbf{R}_{\mathbf{X}})^{\mathcal{U}} = \{r_{(x_1, x_2)} \otimes r_{(x_2, x_3)} \otimes r_{(x_1, x_3)} : (x_1, x_2, x_3) \in X_1 \times X_2 \times X_3\}, \quad (10.30)$$

with its ambient product

$$(\mathbf{R}_{\mathbf{X}})^{[\mathcal{U}]} = R_{X_1 \times X_2} \times R_{X_2 \times X_3} \times R_{X_1 \times X_3}. \quad (10.31)$$

Observe, via Riesz representation and Hahn-Banach, that the dual space of  $C_{(\mathbf{R}_{\mathbf{X}})^{\mathcal{U}}}(\mathbf{\Omega}_{\mathbf{X}^{[\mathcal{U}]}})$  ( $= \{\text{continuous functions on } \mathbf{\Omega}_{\mathbf{X}^{[\mathcal{U}]}} \text{ with spectrum in } (\mathbf{R}_{\mathbf{X}})^{\mathcal{U}}\}$ ) is

$$B((\mathbf{R}_{\mathbf{X}})^{\mathcal{U}}) := \widehat{M}(\mathbf{\Omega}_{\mathbf{X}^{[\mathcal{U}]}}) / \{\hat{\lambda} \in \widehat{M}(\mathbf{\Omega}_{\mathbf{X}^{[\mathcal{U}]}}) : \hat{\lambda} = 0 \text{ on } (\mathbf{R}_{\mathbf{X}})^{\mathcal{U}}\}$$

( $= \{\text{restrictions of Walsh transforms of measures on } M(\mathbf{\Omega}_{\mathbf{X}^{[\mathcal{U}]}}) \text{ to } (\mathbf{R}_{\mathbf{X}})^{\mathcal{U}}\}$ ), and obtain

**Proposition 10.7.**

$$\tilde{\mathcal{V}}_{\mathcal{U}}(\mathbf{X}^{[m]}) = B((\mathbf{R}_{\mathbf{X}})^{\mathcal{U}}).$$

Proposition 9.2 is the instance  $\mathcal{U} = (\{1\}, \dots, \{n\})$ .

**Remark 10.8.**

i. Given infinite sets  $E_1, \dots, E_d$ , and  $F \subset E_1 \times \cdots \times E_d$ , we let

$$\begin{aligned} \Psi_F(s) &:= \max \{ |F \cap (A_1 \times \cdots \times A_d)| : A_i \subset E_i, |A_i| = s, i \in [d] \}, \\ s &= 1, 2, \dots, \end{aligned} \quad (10.32)$$

and define the *combinatorial dimension* of  $F$  (relative to  $E_1 \times \cdots \times E_d$ ) to be

$$\dim F = \limsup_{s \rightarrow \infty} \frac{\log \Psi_F(s)}{\log s}. \quad (10.33)$$

If  $\dim F = \alpha$ , and

$$0 < \liminf_{s \rightarrow \infty} \frac{\Psi_F(s)}{s^\alpha} \leq \limsup_{s \rightarrow \infty} \frac{\Psi_F(s)}{s^\alpha} < \infty, \quad (10.34)$$

then we say that  $F$  is an  $\alpha$ -product.

Given a covering sequence  $\mathcal{U}$  of  $[m]$ , consider the solution to the linear programming problem,

$$\max \left\{ v_1 + \cdots + v_m : \sum_{j \in S} v_j \leq 1, S \in \mathcal{U}, v_i \geq 0, i \in [m] \right\} := \alpha(\mathcal{U}) = \alpha. \quad (10.35)$$

The main result in [9] asserts that if the sets  $X_j$  ( $j \in [m]$ ) are infinite, then  $\mathbf{X}^{\mathcal{U}}$  is an  $\alpha$ -product, and in particular,

$$\dim \mathbf{X}^{\mathcal{U}} = \alpha(\mathcal{U}). \quad (10.36)$$

In general, we have

$$\tilde{\mathcal{V}}_{\mathcal{U}}(\mathbf{X}^{[m]}) = l^\infty(\mathbf{X}^{[m]}) \iff \alpha(\mathcal{U}) = 1. \quad (10.37)$$

Moreover, for all covering sequences  $\mathcal{U}_1$  and  $\mathcal{U}_2$  of  $[m]$ ,

$$\alpha(\mathcal{U}_1) \neq \alpha(\mathcal{U}_2) \implies \tilde{\mathcal{V}}_{\mathcal{U}_1}(\mathbf{X}^{[m]}) \neq \tilde{\mathcal{V}}_{\mathcal{U}_2}(\mathbf{X}^{[m]}). \quad (10.38)$$

For example, if  $\mathcal{U}$  is the sequence in (10.20), then (by inspection)  $\alpha(\mathcal{U}) = 3/2$ , and therefore by (10.37),

$$\tilde{\mathcal{V}}_{\mathcal{U}}(\mathbf{X}^{[3]}) \subsetneq l^\infty(\mathbf{X}^{[3]}). \quad (10.39)$$

The proper inclusion in (10.39) can be proved also directly, by verifying that for every positive integer  $N$ , there exist  $\theta_N \in l^\infty([N]^3)$  such that

$$\|\theta_N\|_\infty = 1,$$

and

$$\|\theta_N\|_{\mathcal{V}_{\mathcal{U}}([N]^3)} \xrightarrow{N \rightarrow \infty} \infty. \quad (10.40)$$

To this end, we use the Kahane-Salem-Zygmund probabilistic estimates (e.g., [22, pp. 68-9]) to produce for every positive integer  $N$ ,

$$\theta_N(i, j, k) := \epsilon_{ijk} \in \{-1, 1\}, \quad (i, j, k) \in [N]^3 \quad (10.41)$$



such that

$$\left\| \sum_{(i,j,k) \in [N]^3} \epsilon_{ijk} r_{ij} \otimes r_{jk} \otimes r_{ik} \right\|_{L^\infty} \leq KN^2, \quad (10.42)$$

wherein Rademacher functions are defined on  $\Omega_{\mathbb{N}^2}$ , and  $K > 0$  is an absolute constant. We then let

$$f_N = \frac{1}{KN^2} \sum_{(i,j,k) \in [N]^3} \epsilon_{ijk} r_{ij} \otimes r_{jk} \otimes r_{ik}, \quad (10.43)$$

and, by applying duality (Proposition 10.7), deduce

$$\sum_{(i,j,k) \in [N]^3} \hat{f}_N(r_{ij} \otimes r_{jk} \otimes r_{ik}) \theta_N(i, j, k) = \frac{N}{K} \leq \|\theta_N\|_{\mathcal{V}_{\mathcal{U}}([N]^3)}, \quad (10.44)$$

thus verifying (10.40).

But more is true: if  $\mathbf{X}^{\mathcal{U}}$  is an  $\alpha$ -product, then

$$l^{\frac{2\alpha}{\alpha-1}}(\mathbf{X}^{[m]}) \subset \tilde{\mathcal{V}}_{\mathcal{U}}(\mathbf{X}^{[m]}), \quad (10.45)$$

and if the  $X_j$  ( $j \in [m]$ ) are infinite, then

$$l^p(\mathbf{X}^{[m]}) \not\subset \tilde{\mathcal{V}}_{\mathcal{U}}(\mathbf{X}^{[m]}), \quad p > \frac{2\alpha}{\alpha-1}, \quad (10.46)$$

which, in particular, implies (10.38). ( $l^p(A)$  denotes the space of all scalar-valued functions  $\mathbf{x}$  on a domain  $A$  such that  $\|\mathbf{x}\|_p := (\sum_{\alpha \in A} |\mathbf{x}(\alpha)|^p)^{\frac{1}{p}} < \infty$ .) In the case of the 3/2-product, (10.46) can be verified directly by slightly altering the proof of (10.39) above.

All this (and more) is detailed in [7, Ch. XII, Ch. XII]; see also the survey article [8].

ii. Suppose  $\mathcal{U} = (S_1, \dots, S_n)$  covers  $[m]$ . If  $f \in \tilde{\mathcal{V}}_n(\mathbf{X}^{[\mathcal{U}]})$ , then the restriction of  $f$  to  $\mathbf{X}^{\mathcal{U}}$  is *a fortiori* in  $\tilde{\mathcal{V}}_{\mathcal{U}}(\mathbf{X}^{[m]})$ , and we have

$$\|f\|_{\tilde{\mathcal{V}}_{\mathcal{U}}(\mathbf{X}^{[m]})} \leq \|f\|_{\tilde{\mathcal{V}}_n(\mathbf{X}^{[\mathcal{U}]})}. \quad (10.47)$$

For example, if  $\mathcal{U} = (\{(1, 2), \{2, 3\}, \{1, 3\})$ , and

$$f \in \tilde{\mathcal{V}}_3((X_1 \times X_2) \times (X_2 \times X_3) \times (X_1 \times X_3)),$$

then by Proposition 9.2, there exists  $\lambda \in M(\Omega_{\mathbf{X}^{[\mathcal{U}]}})$ , where  $\Omega_{\mathbf{X}^{[\mathcal{U}]}}$  is given by (10.29), such that

$$\begin{aligned} f((x_1, x_2), (x_3, x_4), (x_5, x_6)) &= \hat{\lambda}(r_{(x_1, x_2)} \otimes r_{(x_3, x_4)} \otimes r_{(x_5, x_6)}), \\ (x_1, x_2) &\in X_1 \times X_2, \quad (x_3, x_4) \in X_3 \times X_4, \quad (x_5, x_6) \in X_1 \times X_3. \end{aligned} \quad (10.48)$$

Then (obviously!),

$$\begin{aligned} f((x_1, x_2), (x_2, x_3), (x_1, x_3)) &= \hat{\lambda}(r_{(x_1, x_2)} \otimes r_{(x_2, x_3)} \otimes r_{(x_1, x_3)}), \\ (x_1, x_2, x_3) &\in X_1 \times X_2 \times X_3, \end{aligned} \quad (10.49)$$

which, by Proposition 10.7, implies  $f \in \tilde{\mathcal{V}}_{\mathcal{U}}(\mathbf{X}^{[3]})$  and  $\|f\|_{\tilde{\mathcal{V}}_{\mathcal{U}}(\mathbf{X}^{[3]})} \leq \|f\|_{\tilde{\mathcal{V}}_3(\mathbf{X}^{[3]})}$ .

**10.3. A characterization of projectively continuous functionals.** We return to multilinear functionals on a Hilbert space. Let  $m$  be a positive integer, and let

$$\mathcal{U} = (S_1, \dots, S_n)$$

be a covering sequence of  $[m]$ . Given a set  $A$  and  $\theta \in l^\infty(A^m)$ , we define (formally) an  $n$ -linear functional  $\eta_{\mathcal{U}, \theta}$  on

$$l^2(A^{S_1}) \times \dots \times l^2(A^{S_n})$$

by

$$\eta_{\mathcal{U}, \theta}(\mathbf{x}_1, \dots, \mathbf{x}_n) := \sum_{\alpha \in A^m} \theta(\alpha) \mathbf{x}_1(\pi_{S_1}(\alpha)) \cdots \mathbf{x}_n(\pi_{S_n}(\alpha)), \quad (10.50)$$

$$(\mathbf{x}_1, \dots, \mathbf{x}_n) \in l^2(A^{S_1}) \times \dots \times l^2(A^{S_n}).$$

We refer to the functionals defined by (10.50) as multilinear functionals *based on*  $\mathcal{U}$ . Note that the definition of  $\eta_{\mathcal{U}, \theta}$  at this point is completely formal: modes of summation and convergence in (10.50) have not been specified. Denote

$$k_j(\mathcal{U}) := |\{i : j \in S_i\}|, \quad j \in [m], \quad (10.51)$$

(*incidence of  $j$  in  $\mathcal{U}$* ), and

$$I_{\mathcal{U}} := \min\{k_j(\mathcal{U}) : j \in [m]\}. \quad (10.52)$$

**Lemma 10.9** (cf. [5, Lemma 1.2]). *Let  $\mathcal{U} = (S_1, \dots, S_n)$  be a covering sequence of  $[m]$  with  $I_{\mathcal{U}} \geq 2$ , and let  $\theta \in l^\infty(A^m)$ . Then*

$$\sum_{\alpha \in A^m} |\theta(\alpha) \mathbf{x}_1(\pi_{S_1}(\alpha)) \cdots \mathbf{x}_n(\pi_{S_n}(\alpha))| \leq \|\theta\|_\infty,$$

$$(\mathbf{x}_1, \dots, \mathbf{x}_n) \in B_{l^2(A^{S_1})} \times \dots \times B_{l^2(A^{S_n})}.$$

*That is,  $\eta_{\mathcal{U}, \theta}$  is a bounded  $n$ -linear functional on  $l^2(A^{S_1}) \times \dots \times l^2(A^{S_n})$ , and*

$$\|\eta_{\mathcal{U}, \theta}\|_\infty \leq \|\theta\|_\infty.$$

*Proof (by induction on  $m$ ).* It suffices to verify

$$\sum_{\alpha \in A^m} |\mathbf{x}_1(\pi_{S_1}(\alpha)) \cdots \mathbf{x}_n(\pi_{S_n}(\alpha))| \leq \|\mathbf{x}_1\|_2 \cdots \|\mathbf{x}_n\|_2, \quad (10.53)$$

$$(\mathbf{x}_1, \dots, \mathbf{x}_n) \in l^2(A^{S_1}) \times \cdots \times l^2(A^{S_n}).$$

Suppose  $m = 1$ , and

$$\mathcal{U} = (\underbrace{\{1\}, \dots, \{1\}}_n), \quad n = I_{\mathcal{U}} \geq 2.$$

In this case, apply to (10.53) the  $n$ -linear Hölder inequality with conjugate vector  $(\frac{1}{n}, \dots, \frac{1}{n})$ , and then apply  $\|\cdot\|_{I_{\mathcal{U}}} \leq \|\cdot\|_2$  (convexity):

$$\begin{aligned} \sum_{\alpha \in A} |\mathbf{x}_1(\alpha) \cdots \mathbf{x}_n(\alpha)| &\leq \|\mathbf{x}_1\|_n \cdots \|\mathbf{x}_n\|_n \\ &\leq \|\mathbf{x}_1\|_2 \cdots \|\mathbf{x}_n\|_2, \quad (\mathbf{x}_1, \dots, \mathbf{x}_n) \in l^2(A) \times \cdots \times l^2(A). \end{aligned}$$

Now take  $m > 1$ , and let  $\mathcal{U} = (S_1, \dots, S_n)$  be a covering sequence of  $[m]$ . Let

$$S'_i = S_i \setminus \{m\}, \quad i = 1, \dots, n.$$

Then,  $\mathcal{U}' = (S'_1, \dots, S'_n)$  covers  $[m-1]$ , and  $I_{\mathcal{U}'} \geq 2$ . Let  $\mathbf{x}_1 \in l^2(A^{S_1}), \dots, \mathbf{x}_n \in l^2(A^{S_n})$ . Denote

$$T := \{i : m \in S_i\}. \quad (10.54)$$

For  $i \in T$  and  $u \in A$ , define  $\mathbf{x}_{u,i} \in l^2(A^{S'_i})$  by

$$\mathbf{x}_{u,i}(\alpha) = \mathbf{x}_i(\alpha, u), \quad \alpha \in A^{S'_i}.$$

By applying the induction hypothesis, and then (10.53) in the case  $m = 1$  with  $n = |T|$ , we obtain

$$\begin{aligned} \sum_{\alpha \in A^m} |\mathbf{x}_1(\pi_{S_1}(\alpha)) \cdots \mathbf{x}_n(\pi_{S_n}(\alpha))| &= \\ &= \sum_{u \in A} \sum_{\alpha \in A^{[m-1]}} \prod_{i \in [n] \setminus T} |\mathbf{x}_i(\pi_{S'_i}(\alpha))| \prod_{i \in T} |\mathbf{x}_{u,i}(\pi_{S'_i}(\alpha))| \\ &\leq \prod_{i \in [n] \setminus T} \|\mathbf{x}_i\|_2 \sum_{u \in A} \prod_{i \in T} \|\mathbf{x}_{u,i}\|_2 \leq \|\mathbf{x}_1\|_2 \cdots \|\mathbf{x}_n\|_2. \end{aligned}$$

□

**Remark 10.10.**

i. We illustrate the proof of Lemma 10.9 in the case

$$\mathcal{U} = (\{1, 2\}, \{2, 3\}, \{1, 3\}).$$

Taking  $\theta = 1$ , we have

$$\begin{aligned} \eta_{\mathcal{U}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &= \sum_{(\alpha_1, \alpha_2, \alpha_3) \in A^3} \mathbf{x}_1(\alpha_1, \alpha_2) \mathbf{x}_2(\alpha_2, \alpha_3) \mathbf{x}_3(\alpha_1, \alpha_3), \\ \mathbf{x}_j &\in B_{l^2(A^2)}, \quad j = 1, 2, 3. \end{aligned} \quad (10.55)$$

Then, by applying Cauchy-Schwarz three times in succession to the sums over  $\alpha_1, \alpha_2$ , and  $\alpha_3$ , we obtain

$$\begin{aligned} \sum_{\alpha_3 \in A} \sum_{\alpha_2 \in A} \sum_{\alpha_1 \in A} |\mathbf{x}_1(\alpha_1, \alpha_2) \mathbf{x}_2(\alpha_2, \alpha_3) \mathbf{x}_3(\alpha_1, \alpha_3)| &\leq 1, \\ \mathbf{x}_j &\in B_{l^2(A^2)}, \quad j = 1, 2, 3, \end{aligned} \quad (10.56)$$

which implies Lemma 10.9 in this instance.

Given a positive integer  $s$ , and arbitrary  $s$ -sets  $E_j \subset A^2$ ,  $j = 1, 2, 3$  (i.e.,  $|E_j| = s$ ), we put in (10.56)

$$\mathbf{x}_j = \mathbf{1}_{E_j} / \sqrt{s}, \quad j = 1, 2, 3, \quad (10.57)$$

and deduce

$$|\mathbf{A}^{\mathcal{U}} \cap (E_1 \times E_2 \times E_3)| \leq s^{\frac{3}{2}}, \quad (10.58)$$

where, as in (10.21) and (10.22),

$$\begin{aligned} \mathbf{A}^{\mathcal{U}} &= \{((\alpha_1, \alpha_2), (\alpha_2, \alpha_3), (\alpha_1, \alpha_3)) : (\alpha_1, \alpha_2, \alpha_3) \in A^3\} \\ &\subset A^2 \times A^2 \times A^2 = \mathbf{A}^{[\mathcal{U}]}. \end{aligned} \quad (10.59)$$

To obtain that the estimate in (10.58) is asymptotically best possible, simply take  $F \subset A$  to be a  $k$ -set, where  $k$  is an arbitrary positive integer, and  $E_1 = E_2 = E_3 = F^2$ . We take  $s := k^2 = |E_j|$ ,  $j = 1, 2, 3$ , and then have

$$|\mathbf{A}^{\mathcal{U}} \cap (E_1 \times E_2 \times E_3)| = |F^3| = s^{\frac{3}{2}}. \quad (10.60)$$

Recalling the combinatorial gauge  $\Psi$  in (10.32), we have in this case

$$\begin{aligned} \Psi_{\mathbf{A}^{\mathcal{U}}}(s) &= \max \{ |\mathbf{A}^{\mathcal{U}} \cap (E_1 \times E_2 \times E_3)| : s\text{-sets } E_j \subset A^2, j = 1, 2, 3 \} \\ &\quad s = 1, 2, \dots \end{aligned} \quad (10.61)$$

Combining (10.58) and (10.60), we obtain

$$\lim_{s \rightarrow \infty} \frac{\Psi_{\mathbf{A}^{\mathcal{U}}}(s)}{s^{\frac{3}{2}}} = 1, \quad (10.62)$$

which means that  $\mathbf{A}^{\mathcal{U}}$  is a 3/2-product; cf. (10.34). The statement in (10.62) is a special case of the general result in (10.36).

ii. The functionals  $\eta_{\mathcal{U},\theta}$  of Lemma 10.9 can be classified according to their underlying  $\mathcal{U}$  put in standard form. To illustrate, take

$$\mathcal{U} = (\{1, 2\}, \{1\}, \{2\}), \quad (10.63)$$

and let  $\theta \in l^\infty(A^2)$ . The evaluations of the corresponding trilinear functional

$$\eta_{\mathcal{U},\theta}(\mathbf{x}, \mathbf{y}, \mathbf{z}), \quad (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in l^2(A^2) \times l^2(A) \times l^2(A),$$

can be realized as evaluations of  $\eta_{\mathcal{U}',\theta}$  on  $l^2(A^2) \times l^2(A^2) \times l^2(A^2)$ , where

$$\mathcal{U}' = (\{1, 2\}, \{1, 2\}, \{1, 2\}). \quad (10.64)$$

Specifically, for  $\mathbf{x} \in l^2(A)$ , define

$$\tilde{\mathbf{x}}(\alpha_1, \alpha_2) = \begin{cases} \mathbf{x}(\alpha) & \alpha_1 = \alpha_2 \\ 0 & \alpha_1 \neq \alpha_2, \quad (\alpha_1, \alpha_2) \in A^2, \end{cases} \quad (10.65)$$

and then write

$$\eta_{\mathcal{U},\theta}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \eta_{\mathcal{U}',\theta}(\mathbf{x}, \tilde{\mathbf{y}}, \tilde{\mathbf{z}}), \quad (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in l^2(A^2) \times l^2(A) \times l^2(A). \quad (10.66)$$

Next we identify  $\{1, 2\}$  with  $\{1\}$ , and replace  $\mathcal{U}'$  by

$$\mathcal{U}'' = (\{1\}, \{1\}, \{1\}). \quad (10.67)$$

The evaluations of  $\eta_{\mathcal{U},\theta}$  can now be realized as evaluations of a trilinear functional based on  $\mathcal{U}''$ . Specifically, write  $B = A^2$ , and rewrite (10.65) and (10.66) accordingly: For  $\mathbf{x} \in l^2(A^2)$ , let

$$\tilde{\mathbf{x}}(\beta) = \mathbf{x}(\alpha_1, \alpha_2), \quad \beta = (\alpha_1, \alpha_2), \quad (10.68)$$

and for  $\mathbf{x} \in l^2(A)$ , let

$$\tilde{\mathbf{x}}(\beta) = \begin{cases} \mathbf{x}(\alpha) & \text{if } \beta = (\alpha, \alpha) \quad \alpha \in A \\ 0 & \text{if } \beta = (\alpha_1, \alpha_2), \quad \alpha_1 \neq \alpha_2, \quad (\alpha_1, \alpha_2) \in A^2. \end{cases} \quad (10.69)$$

Write

$$\tilde{\theta}(\beta) = \theta(\alpha_1, \alpha_2), \quad \beta = (\alpha_1, \alpha_2). \quad (10.70)$$

Then,

$$\eta_{\mathcal{U},\theta}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \eta_{\mathcal{U}'',\tilde{\theta}}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \tilde{\mathbf{z}}), \quad (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in l^2(A^2) \times l^2(A) \times l^2(A), \quad (10.71)$$

where  $\eta_{\mathcal{U}'',\tilde{\theta}}$  is the trilinear functional defined on  $l^2(B) \times l^2(B) \times l^2(B)$ . Similarly, any trilinear functional based on

$$\mathcal{V} = (\{1, 2, 3\}, \{2, 3\}, \{3\}) \quad (10.72)$$

is "subsumed" by a trilinear functional based also on  $(\{1\}, \{1\}, \{1\})$ ; that is, any trilinear functional based on  $\mathcal{V}$  can be realized as a trilinear functional based on  $\mathcal{U}''$ . In this sense, trilinear functionals based, respectively, on  $\mathcal{U}$  in (10.63) and  $\mathcal{V}$  in (10.72) belong to the same class.

We say that a covering sequence  $\mathcal{U} = (S_1, \dots, S_n)$  of  $[m]$  is in *standard form* (or,  $\mathcal{U}$  is *standard*) if for all  $j$  and  $k$  in  $[n]$ ,

$$S_j \neq S_k \Rightarrow S_j \triangle S_k \neq \emptyset, \quad (10.73)$$

( $\triangle$  = symmetric difference), and if for all  $j$  and  $k$  in  $[m]$  ( $j \neq k$ ), there exists  $S_i \in \mathcal{U}$  such that

$$j \in S_i \text{ and } k \notin S_i. \quad (10.74)$$

Moreover, we consider two standard covering sequences of  $[m]$

$$\mathcal{U} = (S_1, \dots, S_n) \text{ and } \mathcal{V} = (T_1, \dots, T_n)$$

to be equivalent if there exist permutations  $\tau$  of  $[m]$  and  $\sigma$  of  $[n]$  such that  $\tau[S_i] = T_{\sigma(i)}$  for every  $i \in [n]$ . For example,

$$\begin{aligned} \mathcal{U} &= (\{1, 2, 3\}, \{2, 3, 4\}, \{1, 3, 4\}) \\ \text{and} \\ \mathcal{V} &= (\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}) \end{aligned} \quad (10.75)$$

are in standard form, and are equivalent via the cycles  $\tau = (123)$  and  $\sigma = (23)$ .

We say that a covering sequence  $\mathcal{U}$  is *subsumed* by a covering sequence  $\mathcal{V}$ , and write  $\mathcal{U} < \mathcal{V}$ , if every multilinear functional based on  $\mathcal{U}$  can be realized as a multilinear functional based on  $\mathcal{V}$ . (We forego a precise definition.)

Every covering sequence  $\mathcal{U} = (S_1, \dots, S_n)$  of  $[m]$  is subsumed by a standard sequence  $\mathcal{U}''$ . To obtain such  $\mathcal{U}''$ , first let  $\mathcal{U}'$  be a sequence derived from  $\mathcal{U}$  by replacing  $S_j$  with  $S_k$  whenever  $S_j \subsetneq S_k$  for  $j$  and  $k$  in  $[n]$ . Next we consider  $j$  and  $k$  in  $[m]$  to be equivalent (relative to  $\mathcal{U}'$ ) if for all  $S_i \in \mathcal{U}'$ ,

$$j \in S_i \Leftrightarrow k \in S_i, \quad (10.76)$$

and then take  $\{j_1, \dots, j_\ell\} \subset [m]$  to be a list of equivalence class representatives. Now let

$$S''_i = \{k : j_k \in S_i\}, \quad i \in [n], \quad (10.77)$$

and

$$\mathcal{U}'' = (S_1'', \dots, S_n''). \quad (10.78)$$

Then,  $\mathcal{U}''$  is a standard covering sequence of  $[\ell]$ ,

$$\alpha(\mathcal{U}'') = \alpha(\mathcal{U}') = \alpha(\mathcal{U}), \quad I_{\mathcal{U}''} = I_{\mathcal{U}'} \geq I_{\mathcal{U}}, \quad \text{and} \quad \mathcal{U} < \mathcal{U}''.$$

Observe that  $\mathcal{U}''$  need not be unique (even up to equivalence).

For every integer  $n \geq 2$ , there is a finite number  $\mathcal{S}(n)$  of standard covering sequences  $\mathcal{U}$  with  $n$  terms and  $I_{\mathcal{U}} \geq 2$ . For  $n = 2$  we have only  $\mathcal{U} = (\{1\}, \{1\})$ . For  $n = 3$ , we have (up to equivalence)

$$\begin{aligned} \mathcal{U}_1 &= (\{1\}, \{1\}, \{1\}), \\ \mathcal{U}_2 &= (\{1, 2\}, \{2, 3\}, \{1, 3\}), \\ \mathcal{U}_3 &= (\{1, 2, 4\}, \{2, 3, 4\}, \{1, 3, 4\}), \\ \text{and} \\ \mathcal{U}_4 &= (\{1, 2, 4\}, \{2, 3, 4\}, \{1, 3\}), \end{aligned} \quad (10.79)$$

and no more. Note that  $\alpha(\mathcal{U}_1) = 1$ , whereas

$$\alpha(\mathcal{U}_2) = \alpha(\mathcal{U}_3) = \alpha(\mathcal{U}_4) = \frac{3}{2}. \quad (10.80)$$

Note also that  $\mathcal{U}_1 < \mathcal{U}_i$  for  $i = 2, 3, 4$ , and  $\mathcal{U}_2 < \mathcal{U}_i$  for  $i = 3, 4$ . Otherwise,  $\mathcal{S}(n)$  grows rapidly as  $n$  increases.

The theorem below addresses the question: when are the functionals in Lemma 10.9 projectively continuous?

**Theorem 10.11.** *Let  $\mathcal{U} = (S_1, \dots, S_n)$  be a covering sequence of  $[m]$  with  $I_{\mathcal{U}} \geq 2$ . Let  $A$  be an infinite set, and let  $\theta \in l^\infty(A^m)$ .*

*If  $\theta \in \tilde{\mathcal{V}}_{\mathcal{U}}(A^m)$ , then  $\eta_{\mathcal{U}, \theta}$  is projectively continuous.*

*If  $\eta_{\mathcal{U}, \theta}$  is projectively bounded, then  $\theta \in \tilde{\mathcal{V}}_{\mathcal{U}}(A^m)$ , and (therefore)  $\eta_{\mathcal{U}, \theta}$  is projectively continuous.*

*Moreover,*

$$K^{-\beta_{\mathcal{U}}} \|\eta_{\mathcal{U}, \theta}\|_{\tilde{\mathcal{V}}_n(\mathbf{B}^{[U]})} \leq \|\theta\|_{\tilde{\mathcal{V}}_{\mathcal{U}}(A^m)} \leq \|\eta_{\mathcal{U}, \theta}\|_{\tilde{\mathcal{V}}_n(\mathbf{B}^{[U]})}, \quad (10.81)$$

where

$$\begin{aligned} \mathbf{B}^{[U]} &:= B_{l^2(A^{S_1})} \times \cdots \times B_{l^2(A^{S_n})}, \\ \beta_{\mathcal{U}} &:= \sum_{j=1}^m k_j(\mathcal{U}) \quad \left( = \sum_{i=1}^n |S_i| \right), \end{aligned}$$

and  $K > 1$  is the constant in (3.11).

### 11. Proof of Theorem 10.11

The proof has three parts.

**11.1. A multilinear Parseval-like formula.** Let  $m$  be a positive integer, and let  $\mathcal{U} = (S_1, \dots, S_n)$  be a covering sequence of  $[m]$  with  $I_{\mathcal{U}} \geq 2$ . Taking  $\theta = 1$ , we denote

$$\eta_{\mathcal{U}}(\mathbf{x}_1, \dots, \mathbf{x}_n) := \sum_{\alpha \in A^m} \mathbf{x}_1(\pi_{S_1}(\alpha)) \cdots \mathbf{x}_n(\pi_{S_n}(\alpha)), \quad (11.1)$$

$$(\mathbf{x}_1, \dots, \mathbf{x}_n) \in l^2(A^{S_1}) \times \cdots \times l^2(A^{S_n}).$$

We proceed to show that  $\eta_{\mathcal{U}}$  is projectively continuous.

**Lemma 11.1** (cf. Theorem 3.5). *Let  $A$  be an infinite set. For every integer  $k \geq 2$ , there exists a one-one map*

$$\Phi_k : l^2(A) \rightarrow L^\infty(\Omega_A, \mathbb{P}_A),$$

*which is continuous with respect to the  $l^2(A)$ -norm (on its domain) and the  $L^2(\Omega_A, \mathbb{P}_A)$ -norm (on its range), and has the following properties:*

$$\|\Phi_k(\mathbf{x})\|_{L^\infty} \leq K \|\mathbf{x}\|_2, \quad \mathbf{x} \in l^2(A), \quad (11.2)$$

*where  $K > 1$  is an absolute constant;*

$$\Phi_k(c\mathbf{x}) = c\Phi_k(\mathbf{x}), \quad c \in \mathbb{R}, \quad \mathbf{x} \in l^2(A); \quad (11.3)$$

$$\begin{aligned} \sum_{\alpha \in A} \mathbf{x}_1(\alpha) \cdots \mathbf{x}_k(\alpha) &= (\Phi_k(\mathbf{x}_1) * \cdots * \Phi_k(\mathbf{x}_k))(\mathbf{e}) \\ &= \int_{(\Omega_A)^{k-1}} \left( \left( \prod_{j=1}^{k-1} \Phi_k(\mathbf{x}_j)(\omega_j) \right) \Phi_k(\mathbf{x}_k)(\omega_1 \cdots \omega_{k-1}) \right) d\omega_1 \cdots d\omega_{k-1} \\ &= \sum_{\gamma \in \widehat{\Omega}_A} (\widehat{\Phi_k(\mathbf{x}_1)}(\gamma) \cdots \widehat{\Phi_k(\mathbf{x}_k)}(\gamma)), \end{aligned} \quad (11.4)$$

$$\mathbf{x}_1 \in l^2(A), \quad \dots, \quad \mathbf{x}_k \in l^2(A),$$

*where  $d\omega_1 \cdots d\omega_{k-1}$  stands for  $\mathbb{P}_A(d\omega_1) \times \cdots \times \mathbb{P}_A(d\omega_{k-1})$ ;  $*$  denotes convolution over  $\Omega_A$ , and  $\mathbf{e}(\alpha) = 1$  for  $\alpha \in A$ .*



*Proof.* For  $k = 2$ , the map  $\Phi_2$  is the  $\Phi$  supplied by Theorem 3.5. To produce  $\Phi_k$  for  $k > 2$ , simply replace the imaginary  $i$  in (5.9) with  $e^{\frac{\pi i}{k}}$ ; that is, let

$$\Phi_k(\mathbf{x}) = \sum_{j=1}^{\infty} (e^{\frac{\pi i}{k}})^{(j-1)} \|\mathbf{x}^{(j)}\|_2 Q_{A_j}(\sigma \mathbf{x}^{(j)}), \quad \mathbf{x} \in l_{\mathbb{R}}^2(A). \quad (11.5)$$

With  $\Phi_k$  in (11.5) replacing  $\Phi$  in (5.9), the proof runs along the same lines as the proof of Theorem 3.5. (The presence of  $e^{\frac{\pi i}{k}}$  in (11.5) guarantees a representation of

$$(\Phi_k(\mathbf{x}_1) * \cdots * \Phi_k(\mathbf{x}_k))(\mathbf{e}) \quad (11.6)$$

by an alternating series analogous to (5.15), simply because  $(e^{\frac{\pi i}{k}})^k = -1$ .  $\square$

Next, given a covering sequence  $\mathcal{U} = (S_1, \dots, S_n)$  of  $[m]$ , we derive an integral representation of  $\eta_{\mathcal{U}}$  that extends the usual (bilinear) Parseval formula

$$\begin{aligned} (f * g)(\mathbf{e}) &= \int_{\omega \in \Omega_A} f(\omega) g(\omega) \mathbb{P}_A(d\omega) \\ &= \sum_{\gamma \in \Omega_A} \hat{f}(\gamma) \hat{g}(\gamma) \end{aligned} \quad (11.7)$$

$$= \eta_{(\{1\}, \{1\})}(\hat{f}, \hat{g}), \quad (f, g) \in L^2(\Omega_A, \mathbb{P}_A) \times L^2(\Omega_A, \mathbb{P}_A).$$

To this end, for  $j \in [m]$  let

$$\kappa_{\mathcal{U}}(j) = \kappa(j) := \max\{i : j \in S_i\}$$

(the index of the "last" term in  $\mathcal{U}$  that contains  $j$ ), and

$$T_j = \{i : j \in S_i, i < \kappa(j)\}.$$

Recall that  $k_j = k_j(\mathcal{U})$  is the incidence of  $j$  in  $\mathcal{U}$  (defined in (10.51)). Note that  $|T_j| = k_j - 1$ . For  $j \in [m]$ , denote

$$\boldsymbol{\omega}_j := (\omega_{\ell j} : \ell \in T_j) \in \Omega_A^{T_j},$$

and then for  $i \in [n]$ , let

$$\xi_{ij}(\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_m) = \begin{cases} \omega_{ij} & \text{if } i \in T_j \\ \prod_{\ell \in T_j} \omega_{\ell j} & \text{if } i = \kappa(j). \end{cases}$$

Given

$$(f_1, \dots, f_n) \in L^2(\Omega_A^{S_1}, \mathbb{P}_A^{S_1}) \times \dots \times L^2(\Omega_A^{S_n}, \mathbb{P}_A^{S_n}),$$

define

$$\begin{aligned} *_{\mathcal{U}}(f_1, \dots, f_n) := \\ \int_{\omega_1 \in \Omega_A^{T_1}} \left( \dots \left( \int_{\omega_m \in \Omega_A^{T_m}} \left\{ \prod_{i=1}^n f_i(\xi_{ij}(\omega_1, \dots, \omega_m) : j \in S_i) \right\} d\omega_m \right) \dots \right) d\omega_1, \end{aligned} \quad (11.8)$$

where

$$d\omega_j := \prod_{i \in T_j} d\omega_{ij}$$

stands for

$$\mathbb{P}_A^{T_j}(d\omega_j), \quad j = 1, \dots, m.$$

For  $m = 1$  and  $\mathcal{U} = (\overbrace{\{1\}, \dots, \{1\}}^n)$ , the right side of (11.8) is the usual  $n$ -fold convolution of  $f_1 \in L^2(\Omega_A, \mathbb{P}_A), \dots, f_n \in L^2(\Omega_A, \mathbb{P}_A)$ , evaluated at  $\mathbf{e}$  (as in (11.4) with  $n = k$ ). In this instance,

$$*_{\mathcal{U}}(f_1, \dots, f_n) = \sum_{\gamma \in \hat{\Omega}_A} \hat{f}_1(\gamma) \cdots \hat{f}_n(\gamma).$$

In the general case, the iterated integrals in (11.8) form a succession of  $k_j$ -fold convolutions ( $j \in [m]$ ). The lemma below – a "fractional-multilinear" extension of Parseval's formula – can be proved by induction on  $m \geq 1$ . (Proof is omitted.)

**Lemma 11.2.** *Let  $m$  be a positive integer, and let  $\mathcal{U} = (S_1, \dots, S_n)$  be a covering sequence of  $[m]$  with  $I_{\mathcal{U}} \geq 2$ . Then, for  $f_i \in L^2(\Omega_A^{S_i}, \mathbb{P}_A^{S_i})$ ,  $i \in [n]$ ,*

$$*_{\mathcal{U}}(f_1, \dots, f_n) = \sum_{\chi \in \hat{\Omega}_A^m} \hat{f}_1(\pi_{S_1}(\chi)) \cdots \hat{f}_n(\pi_{S_n}(\chi))$$

$$= \eta_{\mathcal{U}}(\hat{f}_1, \dots, \hat{f}_n),$$

where

$$\pi_{S_i} : (\hat{\Omega}_A)^m \rightarrow (\hat{\Omega}_A)^{S_i}$$

are the projections in (10.17) defined by (10.18) with  $X_j = \hat{\Omega}_A$ ,  $j \in [m]$ .

The result below is a multilinear extension of Theorem 3.5. Its proof uses induction in a framework of fractional Cartesian products. In Remark 11.5, we illustrate the proof by running the arguments in the archetypal case of a  $3/2$ -product.

**Theorem 11.3.** *Let  $A$  be an infinite set. For every integer  $\ell > 0$ , and  $\ell$ -tuple*

$$\mathbf{k} = (k_1, \dots, k_\ell) \in \mathbb{N}^\ell$$

*with  $k_i \geq 2$ ,  $i = 1, \dots, \ell$ , there exists a one-one map*

$$\Phi_{\mathbf{k}} : l^2(A^\ell) \rightarrow L^\infty(\Omega_A^\ell, \mathbb{P}_A^\ell), \quad (11.9)$$

*with the following properties:*

(1)  $\Phi_{\mathbf{k}}$  is  $(l^2(A^\ell) \rightarrow L^2(\Omega_A^\ell, \mathbb{P}_A^\ell))$ -continuous, and

$$\|\Phi_{\mathbf{k}}(\mathbf{x})\|_{L^\infty} \leq K^\ell \|\mathbf{x}\|_2, \quad \mathbf{x} \in l^2(A^\ell), \quad (11.10)$$

where  $K > 1$  is the absolute constant in (11.2).

(2) Let  $m$  be a positive integer, and let  $\mathcal{U} = (S_1, \dots, S_m)$  be a covering sequence of  $[m]$  with  $I_{\mathcal{U}} \geq 2$ . Let

$$\mathbf{k}_i = (k_j(\mathcal{U}) : j \in S_i), \quad i \in [m], \quad (11.11)$$

where  $k_j(\mathcal{U})$  is the incidence of  $j$  in  $\mathcal{U}$ , defined in (10.51). Then, for  $\mathbf{x}_i \in l^2(A^{S_i})$ ,  $i \in [m]$ ,

$$\begin{aligned} \eta_{\mathcal{U}}(\mathbf{x}_1, \dots, \mathbf{x}_m) &= *_{\mathcal{U}}(\Phi_{\mathbf{k}_1}(\mathbf{x}_1), \dots, \Phi_{\mathbf{k}_m}(\mathbf{x}_m)), \\ &= \sum_{\chi \in \hat{\Omega}_A^m} \widehat{\Phi_{\mathbf{k}_1}(\mathbf{x}_1)}(\pi_{S_1}(\chi)) \cdots \widehat{\Phi_{\mathbf{k}_m}(\mathbf{x}_m)}(\pi_{S_m}(\chi)). \end{aligned} \quad (11.12)$$

*Proof.* The map

$$\Phi_{\mathbf{k}} : l^2(A^\ell) \rightarrow L^\infty(\Omega_A^\ell, \mathbb{P}_A^\ell)$$

is constructed by successive applications of  $\Phi_{k_i}$ . Given  $\mathbf{k} = (k_1, \dots, k_\ell) \in \mathbb{N}^\ell$  with  $k_i \geq 2$  ( $i = 1, \dots, \ell$ ), we formally define

$$\begin{aligned} \Phi_{\mathbf{k}}(\mathbf{x}) &:= (\Phi_{k_\ell} \circ \cdots \circ \Phi_{k_1})(\mathbf{x}) \\ &:= \Phi_{k_\ell}(\cdots (\Phi_{k_1}(\mathbf{x})) \cdots), \quad \mathbf{x} \in l^2(A^\ell), \end{aligned} \quad (11.13)$$

where the  $\Phi_{k_i}$  of Lemma 11.1 are iteratively applied to "slices" of vectors in  $l^2(A^{\ell-i+1})$ , whose coordinates are elements in  $L^\infty(\Omega_A^{i-1}, \mathbb{P}_A^{i-1})$ .

The construction of  $\Phi_{\mathbf{k}}$  is by induction on  $\ell$ . For  $\ell = 1$  and  $k \geq 2$ , we take the  $\Phi_k$  provided by Lemma 11.1. For  $\ell > 1$ , we assume the  $(\ell - 1)$ -step: that for every  $(\ell - 1)$ -tuple

$$\mathbf{k} = (k_1, \dots, k_{\ell-1}) \in \mathbb{N}^{\ell-1}$$

with  $k_i \geq 2$  ( $i = 1, \dots, \ell - 1$ ), we have

$$\Phi_{\mathbf{k}} : l^2(A^{\ell-1}) \rightarrow L^\infty(\Omega_A^{\ell-1}, \mathbb{P}_A^{\ell-1}), \quad (11.14)$$

such that  $\Phi_{\mathbf{k}}$  is  $(l^2(A^{\ell-1}) \rightarrow L^2(\Omega_A^{\ell-1}, \mathbb{P}_A^{\ell-1}))$ -continuous, and

$$\|\Phi_{\mathbf{k}}(\mathbf{x})\|_{L^\infty} \leq K^{\ell-1} \|\mathbf{x}\|_2, \quad \mathbf{x} \in l^2(A^{\ell-1}), \quad (11.15)$$

where  $K > 1$  is the absolute constant in (11.2). Now let

$$\mathbf{k} = (k_1, \dots, k_\ell) \in \mathbb{N}^\ell,$$

with  $k_i \geq 2$  ( $i = 1, \dots, \ell$ ), and

$$\mathbf{k}' = (k_1, \dots, k_{\ell-1}). \quad (11.16)$$

For  $\mathbf{x} \in l^2(A^\ell)$  and fixed  $\alpha \in A$ , define  $\mathbf{x}^{(\alpha)} \in l^2(A^{\ell-1})$  by

$$\mathbf{x}^{(\alpha)}(\alpha_1, \dots, \alpha_{\ell-1}) = \mathbf{x}(\alpha, \alpha_1, \dots, \alpha_{\ell-1}), \quad (\alpha_1, \dots, \alpha_{\ell-1}) \in l^2(A^{\ell-1}). \quad (11.17)$$

(The definition in (11.17) is temporary:  $\mathbf{x}^{(\alpha)}$  is not the same as  $\mathbf{x}^{(j)}$  defined in (5.4).) We apply the map  $\Phi_{\mathbf{k}'}$  (provided by the  $(\ell - 1)$ -step) to  $\mathbf{x}^{(\alpha)}$ , and thus obtain

$$\|\Phi_{\mathbf{k}'}(\mathbf{x}^{(\alpha)})\|_{L^\infty(\Omega_A^{\ell-1}, \mathbb{P}_A^{\ell-1})} \leq K^{\ell-1} \|\mathbf{x}^{(\alpha)}\|_2. \quad (11.18)$$

Then for almost all  $(\omega_1, \dots, \omega_{\ell-1}) \in (\Omega_A^{\ell-1}, \mathbb{P}_A^{\ell-1})$ , we have

$$\begin{aligned} \sum_{\alpha \in A} |\Phi_{\mathbf{k}'}(\mathbf{x}^{(\alpha)})(\omega_1, \dots, \omega_{\ell-1})|^2 &\leq K^{2(\ell-1)} \sum_{\alpha \in A} \|\mathbf{x}^{(\alpha)}\|_{l^2(A^{\ell-1})}^2 \\ &\leq K^{2(\ell-1)} \|\mathbf{x}\|_{l^2(A^\ell)}^2. \end{aligned} \quad (11.19)$$

We then apply  $\Phi_{k_\ell}$  to  $\Phi_{\mathbf{k}'}(\mathbf{x}^{(\cdot)})(\omega_1, \dots, \omega_{\ell-1})$  (Lemma 11.1), and thus obtain

$$\Phi_{\mathbf{k}}(\mathbf{x}) := \Phi_{k_\ell}(\Phi_{\mathbf{k}'}(\mathbf{x}^{(\cdot)})) \in L^\infty(\Omega_A^\ell, \mathbb{P}_A^\ell), \quad (11.20)$$

with the estimate

$$\|\Phi_{\mathbf{k}}(\mathbf{x})\|_{L^\infty} \leq K^\ell \|\mathbf{x}\|_2. \quad (11.21)$$

The  $(l^2(A^\ell) \rightarrow L^2(\Omega_A^\ell, \mathbb{P}_A^\ell))$ -continuity of  $\Phi_{\mathbf{k}}$  follows also by induction on  $\ell \geq 1$ . The case  $\ell = 1$  is Lemma 11.1. Let  $\ell > 1$ , and assume continuity in the case  $\ell - 1$ . To prove the inductive step, we assume that a sequence  $(\mathbf{x}_j)$  in  $l^2(A^\ell)$  converges to  $\mathbf{x}$  in the  $l^2(A^\ell)$ -norm, and then verify that there is a subsequence  $(\mathbf{x}_{j_i})$  such that  $\Phi_{\mathbf{k}}(\mathbf{x}_{j_i})$  converges to  $\Phi_{\mathbf{k}}(\mathbf{x})$  in the  $L^2(\Omega_A^\ell, \mathbb{P}_A^\ell)$ -norm.

For fixed  $\alpha \in A$ , by the assumption and by the definition in (11.17), we have  $\mathbf{x}_j^{(\alpha)}$  converging to  $\mathbf{x}^{(\alpha)}$  in  $l^2(A^{\ell-1})$ . Therefore, by the induction hypothesis,

$$\Phi_{\mathbf{k}'}(\mathbf{x}_j^{(\alpha)}) \xrightarrow{j \rightarrow \infty} \Phi_{\mathbf{k}'}(\mathbf{x}^{(\alpha)}) \quad \text{in } L^2(\Omega_A^{\ell-1}, \mathbb{P}_A^{\ell-1}), \quad (11.22)$$

with  $\mathbf{k}'$  defined in (11.16). By dominated convergence,

$$\sum_{\alpha \in A} \int_{\omega \in \Omega_A^{\ell-1}} |\Phi_{\mathbf{k}'}(\mathbf{x}_j^{(\alpha)})(\omega) - \Phi_{\mathbf{k}'}(\mathbf{x}^{(\alpha)})(\omega)|^2 \mathbb{P}_A^{\ell-1}(d\omega) \xrightarrow{j \rightarrow \infty} 0, \quad (11.23)$$

and therefore, by interchanging sum and integral,

$$\int_{\omega \in \Omega_A^{\ell-1}} \left( \sum_{\alpha \in A} |\Phi_{\mathbf{k}'}(\mathbf{x}_j^{(\alpha)})(\omega) - \Phi_{\mathbf{k}'}(\mathbf{x}^{(\alpha)})(\omega)|^2 \right) \mathbb{P}_A^{\ell-1}(d\omega) \xrightarrow{j \rightarrow \infty} 0. \quad (11.24)$$

Therefore, there exists a subsequence  $(j_i : i \in \mathbb{N})$  with the property that for almost every  $\omega \in (\Omega_A^{\ell-1}, \mathbb{P}_A^{\ell-1})$ ,

$$\Phi_{\mathbf{k}'}(\mathbf{x}_{j_i}^{(\cdot)})(\omega) \xrightarrow{i \rightarrow \infty} \Phi_{\mathbf{k}'}(\mathbf{x}^{(\cdot)})(\omega) \quad \text{in } l^2(A). \quad (11.25)$$

Therefore, by  $(l^2 \rightarrow L^2)$ -continuity in the case  $\ell = 1$  (Lemma 11.1), for almost every  $\omega \in (\Omega_A^{\ell-1}, \mathbb{P}_A^{\ell-1})$ ,

$$\int_{\omega \in \Omega_A} |\Phi_{k_\ell}(\Phi_{\mathbf{k}'}(\mathbf{x}_{j_i}^{(\cdot)})(\omega)) - \Phi_{k_\ell}(\Phi_{\mathbf{k}'}(\mathbf{x}^{(\cdot)})(\omega))|^2 \mathbb{P}_A(d\omega) \xrightarrow{i \rightarrow \infty} 0. \quad (11.26)$$

Therefore, by dominated convergence and the definition of  $\Phi_{\mathbf{k}}$  (cf. (11.20)), we conclude

$$\|\Phi_{\mathbf{k}}(\mathbf{x}_{j_i}) - \Phi_{\mathbf{k}}(\mathbf{x})\|_{L^2}^2 \xrightarrow{i \rightarrow \infty} 0. \quad (11.27)$$

Next, we prove (11.12) by induction on  $m$  (cf. proof of Lemma 10.9). The case  $m = 1$  is Lemma 11.1. Suppose  $m > 1$ , and that  $\mathcal{U} = (S_1, \dots, S_n)$  is a covering sequence of  $[m]$  with  $I_{\mathcal{U}} \geq 2$ . For  $i \in [n]$ , let

$$S'_i = S_i \setminus \{m\},$$

and

$$\mathbf{k}'_i = (k_j(\mathcal{U}') : j \in S'_i).$$

Then,  $\mathcal{U}' = (S'_1, \dots, S'_n)$  is a covering sequence of  $[m-1]$  with  $I_{\mathcal{U}'} \geq 2$ . Let

$$T := \{i : m \in S_i\}$$

(as per (10.54)), and note that if  $i \notin T$ , then  $S'_i = S_i$  and  $\mathbf{k}'_i = \mathbf{k}_i$ . Let

$$\mathbf{x}_1 \in l^2(A^{S_1}), \dots, \mathbf{x}_n \in l^2(A^{S_n}).$$

For  $i \in T$  and  $u \in A$ , define  $\mathbf{x}_i^{(u)} \in l^2(A^{S'_i})$  by

$$\mathbf{x}_i^{(u)}(\boldsymbol{\alpha}) = \mathbf{x}_i(\boldsymbol{\alpha}, u), \quad \boldsymbol{\alpha} \in A^{S'_i}.$$

(Cf. (11.17).) Then, by the induction hypothesis and Lemma 11.2,

$$\begin{aligned} \eta_{\mathcal{U}}(\mathbf{x}_1, \dots, \mathbf{x}_n) &= \\ &= \sum_{u \in A} \sum_{\boldsymbol{\alpha} \in A^{[m-1]}} \prod_{i \in [n] \setminus T} \mathbf{x}_i(\pi_{S'_i}(\boldsymbol{\alpha})) \prod_{i \in T} \mathbf{x}_i^{(u)}(\pi_{S'_i}(\boldsymbol{\alpha})) \\ &= \sum_{u \in A} \sum_{\boldsymbol{\chi} \in \hat{\Omega}_A^{m-1}} \prod_{i \in [n] \setminus T} (\Phi_{\mathbf{k}_i}(\mathbf{x}_i))^\wedge (\pi_{S_i}(\boldsymbol{\chi})) \prod_{i \in T} (\Phi_{\mathbf{k}'_i}(\mathbf{x}_i^{(u)}))^\wedge (\pi_{S'_i}(\boldsymbol{\chi})) \\ &= \sum_{\boldsymbol{\chi} \in \hat{\Omega}_A^{m-1}} \prod_{i \in [n] \setminus T} (\Phi_{\mathbf{k}_i}(\mathbf{x}_i))^\wedge (\pi_{S_i}(\boldsymbol{\chi})) \sum_{u \in A} \prod_{i \in T} (\Phi_{\mathbf{k}'_i}(\mathbf{x}_i^{(u)}))^\wedge (\pi_{S'_i}(\boldsymbol{\chi})). \end{aligned} \tag{11.28}$$

For  $\boldsymbol{\chi} \in \hat{\Omega}_A^{m-1}$  and  $i \in T$ , define (for typographical convenience)  $\mathbf{v}_{i,\boldsymbol{\chi}} \in l^2(A)$  by

$$\mathbf{v}_{i,\boldsymbol{\chi}}(u) = \left( \Phi_{\mathbf{k}'_i}(\mathbf{x}_i^{(u)}) \right)^\wedge (\pi_{S'_i}(\boldsymbol{\chi})).$$

From Lemma 11.1 and the recursive definition of  $\Phi_{\mathbf{k}_i}$ , we obtain

$$\begin{aligned} \sum_{u \in A} \prod_{i \in T} \left( \Phi_{\mathbf{k}'_i}(\mathbf{x}_i^{(u)}) \right)^\wedge (\pi_{S'_i}(\boldsymbol{\chi})) &= \sum_{u \in A} \prod_{i \in T} \mathbf{v}_{i,\boldsymbol{\chi}}(u) \\ &= \sum_{\gamma \in \hat{\Omega}_A} \prod_{i \in T} \left( \Phi_{k_m}(\mathbf{v}_{i,\boldsymbol{\chi}}) \right)^\wedge (\gamma) \\ \text{(by Lemma 11.1, with } k &= k_m := k_m(\mathcal{U}) = |T|) \\ &= \sum_{\gamma \in \hat{\Omega}_A} \prod_{i \in T} (\Phi_{\mathbf{k}_i}(\mathbf{x}_i))^\wedge (\pi_{S_i}((\boldsymbol{\chi}, \gamma))) \\ \text{(by the recursive definition of } &\Phi_{\mathbf{k}_i}). \end{aligned} \tag{11.29}$$

Substituting (11.29) in (11.28), we obtain

$$\begin{aligned} \eta_{\mathcal{U}}(\mathbf{x}_1, \dots, \mathbf{x}_n) &= \sum_{\boldsymbol{\chi} \in \hat{\Omega}_A^{m-1}} \prod_{i \in [n] \setminus T} (\Phi_{\mathbf{k}_i}(\mathbf{x}_i))^{\wedge} (\pi_{S_i}(\boldsymbol{\chi})) \sum_{\gamma \in \hat{\Omega}_A} \prod_{i \in T} (\Phi_{\mathbf{k}_i}(\mathbf{x}_i))^{\wedge} (\pi_{S_i}(\boldsymbol{\chi}, \gamma)) \\ &= \sum_{\boldsymbol{\chi} \in \hat{\Omega}_A^m} \prod_{i=1}^n (\Phi_{\mathbf{k}_i}(\mathbf{x}_i))^{\wedge} (\pi_{S_i}(\boldsymbol{\chi})), \end{aligned}$$

and thus the integral representation in (11.12).  $\square$

**Corollary 11.4.** *For every integer  $m \geq 1$ , and every covering sequence  $\mathcal{U} = (S_1, \dots, S_n)$  of  $[m]$  with  $I_{\mathcal{U}} \geq 2$ , the multilinear functional  $\eta_{\mathcal{U}}$  (defined in (11.1)) is projectively continuous, and*

$$\|\eta_{\mathcal{U}}\|_{\hat{V}_n(\mathbf{B}^{[\mathcal{U}]})} \leq K^{\beta_{\mathcal{U}}}, \quad (11.30)$$

where

$$\begin{aligned} \mathbf{B}^{[\mathcal{U}]} &:= B_{l^2(A^{S_1})} \times \dots \times B_{l^2(A^{S_n})}, \\ \beta_{\mathcal{U}} &:= \sum_{j=1}^m k_j(\mathcal{U}) \quad \left( = \sum_{i=1}^n |S_i| \right), \end{aligned} \quad (11.31)$$

and  $K > 1$  is the constant in (3.11).

**Remark 11.5.** We illustrate the proof of Theorem 11.3 in the case  $m = 3$  and

$$\mathcal{U} = (\{1, 2\}, \{2, 3\}, \{1, 3\}),$$

whence  $\ell = 2$ , and

$$\begin{aligned} \mathbf{k}_1 &= (k_1(\mathcal{U}), k_2(\mathcal{U})) = (2, 2) \\ \mathbf{k}_2 &= (k_2(\mathcal{U}), k_3(\mathcal{U})) = (2, 2) \\ \mathbf{k}_3 &= (k_1(\mathcal{U}), k_3(\mathcal{U})) = (2, 2). \end{aligned} \quad (11.32)$$

(Cf. (11.11).)

First we construct the map

$$\Phi_{(2,2)} : l^2(A^2) \rightarrow L^\infty(\Omega_A^2, \mathbb{P}_A^2). \quad (11.33)$$

For  $\mathbf{x} \in l^2(A^2)$  and fixed  $\alpha \in A$ , we apply  $\Phi_2$  (supplied by Lemma 11.1 with  $k = 2$ ) to  $\mathbf{x}^{(\alpha)} := \mathbf{x}(\alpha, \cdot) \in l^2(A)$ , and deduce that for almost all  $\omega \in (\Omega_A, \mathbb{P}_A)$

$$\sum_{\alpha \in A} \|\Phi_2(\mathbf{x}^{(\alpha)})(\omega)\|_{l^2(A)}^2 \leq \sum_{\alpha \in A} K^2 \|\mathbf{x}^{(\alpha)}\|_2^2 = K^2 \|\mathbf{x}\|_2^2. \quad (11.34)$$

(Cf. (11.19).) Now apply  $\Phi_2$  to  $\Phi_2(\mathbf{x}^{(\cdot)})(\omega) \in l^2(A)$  for almost all  $\omega \in (\Omega_A, \mathbb{P}_A)$ , and obtain

$$\Phi_{(2,2)}(\mathbf{x}) := \Phi_2(\Phi_2(\mathbf{x}^{(\cdot)})) \in L^\infty(\Omega_A^2, \mathbb{P}_A^2), \quad (11.35)$$

with the estimate

$$\|\Phi_{(2,2)}(\mathbf{x})\|_{L^\infty} \leq K \operatorname{ess\,sup}_{\omega \in \Omega_A} \|\Phi_2(\mathbf{x}^{(\cdot)})(\omega)\|_{l^2(A)} \leq K^2 \|\mathbf{x}\|_2. \quad (11.36)$$

(Cf. (11.20) and (11.21).)

Next, to verify the  $(l^2 \rightarrow L^2)$ -continuity of  $\Phi_{(2,2)}$ , we prove that if  $(\mathbf{x}_j)$  is a sequence in  $l^2(A^2)$  converging to  $\mathbf{x}$  in the  $l^2(A^2)$ -norm, then there is a subsequence  $(\mathbf{x}_{j_i})$  such that  $\Phi_{(2,2)}(\mathbf{x}_{j_i})$  converges to  $\Phi_{(2,2)}(\mathbf{x})$  in the  $L^2(\Omega_A^2, \mathbb{P}_A^2)$ -norm. First, because  $\mathbf{x}_j^{(\alpha)}$  converges to  $\mathbf{x}^{(\alpha)}$  in the  $l^2(A)$ -norm for every  $\alpha \in l^2(A)$ , we obtain from the  $(l^2 \rightarrow L^2)$ -continuity of  $\Phi_2$ ,

$$\int_{\omega \in \Omega_A} |\Phi_2(\mathbf{x}_j^{(\alpha)})(\omega) - \Phi_2(\mathbf{x}^{(\alpha)})(\omega)|^2 \mathbb{P}_A(d\omega) \xrightarrow{j \rightarrow \infty} 0. \quad (11.37)$$

Therefore, by dominated convergence (via (11.34)), and by interchange of sum and integral, we obtain

$$\int_{\omega \in \Omega_A} \left( \sum_{\alpha \in A} |\Phi_2(\mathbf{x}_j^{(\alpha)})(\omega) - \Phi_2(\mathbf{x}^{(\alpha)})(\omega)|^2 \right) \mathbb{P}_A(d\omega) \xrightarrow{j \rightarrow \infty} 0. \quad (11.38)$$

Therefore, there exists a subsequence  $(j_i)$  such that for almost all  $\omega \in (\Omega_A, \mathbb{P}_A)$

$$\sum_{\alpha \in A} |\Phi_2(\mathbf{x}_{j_i}^{(\alpha)})(\omega) - \Phi_2(\mathbf{x}^{(\alpha)})(\omega)|^2 \xrightarrow{i \rightarrow \infty} 0. \quad (11.39)$$

Therefore, by a second application of the  $(l^2 \rightarrow L^2)$ -continuity of  $\Phi_2$ , we have for almost all  $\omega_2 \in (\Omega_A, \mathbb{P}_A)$

$$\int_{\omega_1 \in \Omega_A} |\Phi_2(\Phi_2(\mathbf{x}_{j_i}^{(\cdot)})(\omega_2))(\omega_1) - \Phi_2(\Phi_2(\mathbf{x}^{(\cdot)})(\omega_2))(\omega_1)|^2 \mathbb{P}_A(d\omega_1) \xrightarrow{i \rightarrow \infty} 0, \quad (11.40)$$

and by dominated convergence we conclude

$$\begin{aligned} & \|\Phi_{(2,2)}(\mathbf{x}_{j_i}) - \Phi_{(2,2)}(\mathbf{x})\|_{L^2}^2 \\ &= \int_{\omega_2 \in \Omega_A} \left( \int_{\omega_1 \in \Omega_A} |\Phi_2(\Phi_2(\mathbf{x}_{j_i}^{(\cdot)})(\omega_2))(\omega_1) - \Phi_2(\Phi_2(\mathbf{x}^{(\cdot)})(\omega_2))(\omega_1)|^2 \mathbb{P}_A(d\omega_1) \right) \mathbb{P}_A(d\omega_2) \xrightarrow{i \rightarrow \infty} 0. \end{aligned} \quad (11.41)$$



Finally, we verify (11.12) by three successive applications of (11.4) with  $k = 2$  (in Lemma 11.1). Specifically, for  $\mathbf{x}, \mathbf{y}, \mathbf{z}$  in  $l^2(A^2)$ ,

$$\begin{aligned}
\eta_{\mathcal{U}}(\mathbf{x}, \mathbf{y}, \mathbf{z}) &= \sum_{\alpha_1, \alpha_2, \alpha_3} \mathbf{x}(\alpha_1, \alpha_2) \mathbf{y}(\alpha_2, \alpha_3) \mathbf{z}(\alpha_1, \alpha_3) \\
&= \sum_{\alpha_3} \sum_{\alpha_2} \mathbf{y}(\alpha_2, \alpha_3) \sum_{\alpha_1} \mathbf{x}(\alpha_1, \alpha_2) \mathbf{z}(\alpha_1, \alpha_3) \\
&= \sum_{\alpha_3} \sum_{\alpha_2} \mathbf{y}(\alpha_2, \alpha_3) \int_{\omega_1} \Phi_2(\mathbf{x}(\cdot, \alpha_2))(\omega_1) \Phi_2(\mathbf{z}(\cdot, \alpha_3))(\omega_1) d\omega_1 \\
&= \sum_{\alpha_3} \int_{\omega_1} \left( \Phi_2(\mathbf{z}(\cdot, \alpha_3))(\omega_1) \sum_{\alpha_2} \mathbf{y}(\alpha_2, \alpha_3) \Phi_2(\mathbf{x}(\cdot, \alpha_2))(\omega_1) \right) d\omega_1 \\
&= \sum_{\alpha_3} \int_{\omega_1} \Phi_2(\mathbf{z}(\cdot, \alpha_3))(\omega_1) \left( \int_{\omega_2} \Phi_2(\mathbf{y}(\cdot, \alpha_3))(\omega_2) \Phi_{(2,2)}(\mathbf{x})(\omega_1, \omega_2) d\omega_2 \right) d\omega_1 \\
&= \int_{\omega_1} \int_{\omega_2} \Phi_{(2,2)}(\mathbf{x})(\omega_1, \omega_2) \left( \sum_{\alpha_3} \Phi_2(\mathbf{y}(\cdot, \alpha_3))(\omega_2) \Phi_2(\mathbf{z}(\cdot, \alpha_3))(\omega_1) \right) d\omega_2 d\omega_1 \\
&= \int_{\omega_1} \int_{\omega_2} \Phi_{(2,2)}(\mathbf{x})(\omega_1, \omega_2) \left( \int_{\omega_3} \Phi_{(2,2)}(\mathbf{y})(\omega_2, \omega_3) \Phi_{(2,2)}(\mathbf{z})(\omega_1, \omega_3) d\omega_3 \right) d\omega_2 d\omega_1 \\
&= \int_{\omega_1} \int_{\omega_2} \int_{\omega_3} \Phi_{(2,2)}(\mathbf{x})(\omega_1, \omega_2) \Phi_{(2,2)}(\mathbf{y})(\omega_2, \omega_3) \Phi_{(2,2)}(\mathbf{z})(\omega_1, \omega_3) d\omega_3 d\omega_2 d\omega_1 \\
&= *_{\mathcal{U}}(\Phi_{(2,2)}(\mathbf{x}), \Phi_{(2,2)}(\mathbf{y}), \Phi_{(2,2)}(\mathbf{z})),
\end{aligned} \tag{11.42}$$

which is the needed integral representation of  $\eta_{\mathcal{U}}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ .

Therefore (cf. Corollary 11.4),

$$\|\eta_{\mathcal{U}}\|_{\tilde{V}_3(B_{l_2^3}^3)} \leq K^{\beta_{\mathcal{U}}} = K^6. \tag{11.43}$$

**11.2. The left-side inequality in (10.81).** To verify sufficiency in Theorem 10.11, we start with  $\theta \in \tilde{\mathcal{V}}_{\mathcal{U}}(A^m)$ . By Proposition 10.7, for every  $\epsilon > 0$  there exist

$$\mu \in M(\Omega_{A^{S_1}} \times \cdots \times \Omega_{A^{S_n}})$$

such that

$$\theta(\alpha) = \hat{\mu}(r_{\pi_{S_1}}(\alpha) \otimes \cdots \otimes r_{\pi_{S_n}}(\alpha)), \quad \alpha \in A^m,$$

and

$$\|\mu\|_M \leq \|\theta\|_{\tilde{\mathcal{V}}_{\mathcal{U}}} + \epsilon. \quad (11.44)$$

For  $\mathbf{x}_1 \in l^2(A^{S_1}), \dots, \mathbf{x}_n \in l^2(A^{S_n})$ , denote

$$G(\mathbf{x}_1, \dots, \mathbf{x}_n) = \sum_{\alpha \in A^m} \mathbf{x}_1(\pi_{S_1}(\alpha)) \cdots \mathbf{x}_n(\pi_{S_n}(\alpha)) r_{\pi_{S_1}}(\alpha) \otimes \cdots \otimes r_{\pi_{S_n}}(\alpha). \quad (11.45)$$

By Lemma 10.9,  $G(\mathbf{x}_1, \dots, \mathbf{x}_n)$  is a continuous function on  $\Omega_{A^{S_1}} \times \cdots \times \Omega_{A^{S_n}}$  with absolutely convergent Walsh series. Therefore, by Parseval's formula (4.2),

$$\int_{\zeta \in \Omega_{\mathbf{A}^{[\mathcal{U}]}}} G(\mathbf{x}_1, \dots, \mathbf{x}_n)(\zeta) \mu(d\zeta) = \eta_{\mathcal{U}, \theta}(\mathbf{x}_1, \dots, \mathbf{x}_n), \quad (11.46)$$

where

$$\Omega_{\mathbf{A}^{[\mathcal{U}]}} := \Omega_{A^{S_1}} \times \cdots \times \Omega_{A^{S_n}}.$$

For a set  $B$ ,  $\mathbf{x} \in l^2(B)$ , and  $\zeta \in \Omega_B$  ( $= \{-1, 1\}^B$ ), define  $\mathbf{x} \bullet \zeta \in l^2(B)$  by

$$(\mathbf{x} \bullet \zeta)(\beta) := \mathbf{x}(\beta) \zeta(\beta) \quad (= \mathbf{x}(\beta) r_{\beta}(\zeta)), \quad \beta \in B, \quad (11.47)$$

whence (obviously!)

$$\|\mathbf{x} \bullet \zeta\|_2 = \|\mathbf{x}\|_2. \quad (11.48)$$

Then by (11.12) (the case  $\theta = 1$ ), for all

$$\zeta = (\zeta_1, \dots, \zeta_n) \in \Omega_{A^{S_1}} \times \cdots \times \Omega_{A^{S_n}} = \Omega_{\mathbf{A}^{[\mathcal{U}]}},$$

we have

$$G(\mathbf{x}_1, \dots, \mathbf{x}_n)(\zeta) = *_U(\Phi_{\mathbf{k}_1}(\mathbf{x}_1 \bullet \zeta_1), \dots, \Phi_{\mathbf{k}_n}(\mathbf{x}_n \bullet \zeta_n)). \quad (11.49)$$

Substituting (11.49) in (11.46), we obtain

$$\eta_{\mathcal{U}, \theta}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \int_{\zeta \in \Omega_{\mathbf{A}^{[\mathcal{U}]}}} \left( *_U(\Phi_{\mathbf{k}_1}(\mathbf{x}_1 \bullet \zeta_1), \dots, \Phi_{\mathbf{k}_n}(\mathbf{x}_n \bullet \zeta_n)) \right) \mu(d\zeta), \quad (11.50)$$

which implies the desired integral representation of  $\eta_{\mathcal{U}, \theta}$ . Combining (11.30), (11.44) and (11.48), we obtain the left side inequality in (10.81).

**Remark 11.6.** To illustrate ideas, we run the proof of the left side of (10.81) in the archetypal case

$$\mathcal{U} = (\{1, 2\}, \{2, 3\}, \{1, 3\}).$$

Let  $\theta \in \tilde{\mathcal{V}}_{\mathcal{U}}(A^3)$ , and (by Proposition 10.7) obtain a measure  $\mu \in M(\Omega_{A^2} \times \Omega_{A^2} \times \Omega_{A^2})$  such that

$$\theta(\alpha_1, \alpha_2, \alpha_3) = \hat{\mu}(r_{(\alpha_1, \alpha_2)} \otimes r_{(\alpha_2, \alpha_3)} \otimes r_{(\alpha_1, \alpha_3)}), \quad (\alpha_1, \alpha_2, \alpha_3) \in A^3. \quad (11.51)$$

Then by Parseval's formula, for  $\mathbf{x}, \mathbf{y}, \mathbf{z}$  in  $l^2(A^2)$ ,

$$\begin{aligned} \eta_{\mathcal{U}, \theta}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \\ \int_{(\Omega_{A^2})^3} \left( \sum_{(\alpha_1, \alpha_2, \alpha_3) \in A^3} \mathbf{x}(\alpha_1, \alpha_2) \mathbf{y}(\alpha_2, \alpha_3) \mathbf{z}(\alpha_1, \alpha_3) r_{(\alpha_1, \alpha_2)} \otimes r_{(\alpha_2, \alpha_3)} \otimes r_{(\alpha_1, \alpha_3)} \right) d\mu = \end{aligned} \quad (11.52)$$

$$\int_{(\Omega_{A^2})^3} \left( \eta_{\mathcal{U}}(\mathbf{x} \bullet \zeta_1, \mathbf{y} \bullet \zeta_2, \mathbf{z} \bullet \zeta_3) \right) \mu(d\zeta_1, d\zeta_2, d\zeta_3),$$

where  $\mathbf{x} \bullet \zeta, \mathbf{y} \bullet \zeta, \mathbf{z} \bullet \zeta$  are the "random" vectors in  $l^2(A^2)$ ,  $\zeta \in \Omega_{A^2}$ , as defined in (11.47). Then, by applying (11.42) (the case  $\theta = 1$ ) to the integrand in (11.52), we deduce

$$\eta_{\mathcal{U}, \theta}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \int_{(\zeta_1, \zeta_2, \zeta_3) \in \Omega_{A^2}} \left( \ast_{\mathcal{U}}(\Phi_{(2,2)}(\mathbf{x} \bullet \zeta_1), \Phi_{(2,2)}(\mathbf{y} \bullet \zeta_2), \Phi_{(2,2)}(\mathbf{z} \bullet \zeta_3)) \right) \mu(d\zeta_1, d\zeta_2, d\zeta_3). \quad (11.53)$$

**11.3. The right-side inequality in (10.81).** To establish necessity in Theorem 10.11, we verify

$$\|\theta\|_{\tilde{\mathcal{V}}_{\mathcal{U}}(A^m)} \leq \|\eta_{\mathcal{U}, \theta}\|_{\tilde{\mathcal{V}}_n(\mathbf{B}^{[\mathcal{U}]})}, \quad (11.54)$$

where

$$\mathbf{B}^{[\mathcal{U}]} := B_{l^2(A^{S_1})} \times \cdots \times B_{l^2(A^{S_n})}.$$

For a set  $E$ , let  $\hat{E} = \{\hat{e}\}_{e \in E}$  denote the standard basis of  $l^2(E)$ ; that is, for  $e \in E$  and  $e' \in E$ ,

$$\hat{e}(e') = \begin{cases} 1 & \text{if } e = e' \\ 0 & \text{if } e \neq e'. \end{cases}$$

Specifically in our setting, for  $S \subset [m]$  and  $\alpha = (\alpha_j : j \in S) \in A^S$ , we have

$$\hat{\alpha} = (\hat{\alpha}_j : j \in S);$$

that is, for  $\boldsymbol{\alpha}' = (\alpha'_j : j \in S) \in A^S$ ,

$$\begin{aligned}\hat{\boldsymbol{\alpha}}(\boldsymbol{\alpha}') &= \prod_{j \in S} \hat{\alpha}_j(\alpha'_j) \\ &= \begin{cases} 1 & \text{if } \boldsymbol{\alpha} = \boldsymbol{\alpha}' \\ 0 & \text{if } \boldsymbol{\alpha} \neq \boldsymbol{\alpha}'. \end{cases}\end{aligned}$$

Restricting  $\eta_{\mathcal{U}, \theta}$  to  $\hat{A}^{S_1} \times \cdots \times \hat{A}^{S_n} := \hat{\mathbf{A}}^{[\mathcal{U}]}$ , we have

$$\|\eta_{\mathcal{U}, \theta}\|_{\tilde{\mathcal{V}}_n(\hat{\mathbf{A}}^{[\mathcal{U}]})} \leq \|\eta_{\mathcal{U}, \theta}\|_{\tilde{\mathcal{V}}_n(\mathbf{B}^{[\mathcal{U}]})}. \quad (11.55)$$

For  $\hat{\boldsymbol{\alpha}} = (\hat{\alpha}_1, \dots, \hat{\alpha}_n) \in \hat{\mathbf{A}}^{[\mathcal{U}]}$ , we have

$$\begin{aligned}\eta_{\mathcal{U}, \theta}(\hat{\boldsymbol{\alpha}}) &= \sum_{\boldsymbol{\alpha}' \in A^m} \theta(\boldsymbol{\alpha}') \hat{\alpha}_1(\pi_{S_1}(\boldsymbol{\alpha}')) \cdots \hat{\alpha}_n(\pi_{S_n}(\boldsymbol{\alpha}')) \\ &= \begin{cases} \theta(\boldsymbol{\alpha}) & \text{if } \boldsymbol{\alpha} \in \mathbf{A}^{\mathcal{U}} \\ 0 & \text{if } \boldsymbol{\alpha} \notin \mathbf{A}^{\mathcal{U}}, \end{cases}\end{aligned} \quad (11.56)$$

where  $\mathbf{A}^{\mathcal{U}}$  is the fractional Cartesian product in (10.19) with  $A$  in place of  $X_i$ ,  $i \in [n]$ . Combining (11.56) and (11.55), we conclude (via (10.47) in Remark 10.8.ii)

$$\|\theta\|_{\tilde{\mathcal{V}}_{\mathcal{U}}(A^m)} = \|\eta_{\mathcal{U}, \theta}\|_{\tilde{\mathcal{V}}_{\mathcal{U}}(\hat{A}^m)} \leq \|\eta_{\mathcal{U}, \theta}\|_{\tilde{\mathcal{V}}_n(\hat{\mathbf{A}}^{[\mathcal{U}]})} \leq \|\eta_{\mathcal{U}, \theta}\|_{\tilde{\mathcal{V}}_n(\mathbf{B}^{[\mathcal{U}]})}. \quad (11.57)$$

**Remark 11.7.** For  $n \geq 3$ , except for covering sequences of type  $(\overbrace{\{1\}, \dots, \{1\}}^n)$ , all other covering sequences  $\mathcal{U} = (S_1, \dots, S_n)$  with  $I_{\mathcal{U}} \geq 2$ , satisfy  $\alpha(\mathcal{U}) > 1$ . For such  $\mathcal{U}$ , by Theorem 10.11, there exist bounded  $n$ -linear functionals  $\eta_{\mathcal{U}, \theta}$  that are not projectively bounded; see Remark 10.8.i and (10.37) therein.

For example, if  $n = 3$ , then we have four types of trilinear functionals  $\eta_{\mathcal{U},\theta}$  ( $i = 1, 2, 3, 4$ ), based on the four standard covering sequences

$$\begin{aligned}\mathcal{U}_1 &= (\{1\}, \{1\}, \{1\}), \\ \mathcal{U}_2 &= (\{1, 2\}, \{2, 3\}, \{1, 3\}), \\ \mathcal{U}_3 &= (\{1, 2, 4\}, \{2, 3, 4\}, \{1, 3, 4\}), \\ \text{and} \\ \mathcal{U}_4 &= (\{1, 2, 4\}, \{2, 3, 4\}, \{1, 3\}).\end{aligned}$$

(See Remark 10.10.ii, and (10.79) therein.) By (10.37), by Theorem 10.11, and because  $\alpha(\mathcal{U}_1) = 1$ , the trilinear functional  $\eta_{\mathcal{U}_1,\theta}$  is projectively continuous for every  $\theta \in l^\infty(A)$ , whereas by (10.80), there exist bounded trilinear functionals  $\eta_{\mathcal{U}_i,\theta}$  ( $i = 2, 3, 4$ ) that are not projectively bounded. Bounded trilinear functionals based on  $\mathcal{U}_2$  that were *not* projectively bounded appeared first in [42].

**11.4. Multilinear extensions of the Grothendieck inequality.** The "averaging" argument used to verify (1.5)  $\Rightarrow$  (1.2) in Proposition 1.1 can be analogously used, via the left-side of (10.81), to prove the following multilinear extension of the Grothendieck inequality. For every integer  $n \geq 2$  and  $m \geq 1$ , and every covering sequence  $\mathcal{U} = (S_1, \dots, S_n)$  of  $[m]$  with  $I_{\mathcal{U}} \geq 2$ , there is a constant  $\mathcal{K}_{\mathcal{U}}$ ,

$$1 < \mathcal{K}_{\mathcal{U}} \leq K^{\sum_{i=1}^n |S_i|}, \quad (11.58)$$

where  $K > 1$  is the constant in (3.11), such that for all  $\theta \in \tilde{\mathcal{V}}_{\mathcal{U}}(A^m)$  and every finitely supported scalar  $n$ -array  $(a_{j_1 \dots j_n})_{(j_1, \dots, j_n) \in \mathbb{N}^n}$ ,

$$\begin{aligned} & \sup \left\{ \left| \sum_{j_1, \dots, j_n} a_{j_1 \dots j_n} \eta_{\mathcal{U},\theta}(\mathbf{x}_{1,j_1}, \dots, \mathbf{x}_{n,j_n}) \right| : (\mathbf{x}_{1,j_1}, \dots, \mathbf{x}_{n,j_n}) \in B_{l^2(A^{S_1})} \times \dots \times B_{l^2(A^{S_n})} \right\} \\ & \leq \mathcal{K}_{\mathcal{U}} \|\theta\|_{\tilde{\mathcal{V}}_{\mathcal{U}}(A^m)} \sup \left\{ \left| \sum_{j_1, \dots, j_n} a_{j_1 \dots j_n} s_{1,j_1} \dots s_{n,j_n} \right| : (s_{1,j_1}, \dots, s_{n,j_n}) \in [-1, 1]^n \right\}. \end{aligned} \quad (11.59)$$

The inequality in (11.59), derived here as a consequence of

$$\|\eta_{\mathcal{U},\theta}\|_{\tilde{\mathcal{V}}_n(\mathbf{B}^{[U]})} \leq K^{\sum_{i=1}^n |S_i|} \|\theta\|_{\tilde{\mathcal{V}}_{\mathcal{U}}(A^m)}, \quad (11.60)$$

seems weaker than the left-side inequality in (10.81). Namely, by applying the duality

$$\left( C_{\mathbf{R}_{\mathbf{X}^{[n]}}}(\Omega_{\mathbf{X}^{[n]}}) \right)^* = B(\mathbf{R}_{\mathbf{X}^{[n]}}) = \tilde{\mathcal{V}}_n(\mathbf{X}^{[n]}), \quad (11.61)$$

$$\text{with } \mathbf{X}^{[n]} = B_{l^2(A^{S_1})} \times \cdots \times B_{l^2(A^{S_n})} := \mathbf{B}^{[\mathcal{U}]},$$

(cf. (9.3), (9.4), Proposition 9.2), we obtain that the multilinear inequality in (11.59) is equivalent to the existence of a complex measure

$$\lambda \in M(\Omega_{B_{l^2(A^{S_1})}} \times \cdots \times \Omega_{B_{l^2(A^{S_n})}})$$

with  $\|\lambda\|_M = \mathcal{K}_{\mathcal{U}} \|\theta\|_{\tilde{\mathcal{V}}_{\mathcal{U}}(A^m)}$ , such that

$$\begin{aligned} \eta_{\mathcal{U},\theta}(\mathbf{x}_1, \dots, \mathbf{x}_n) &= \int_{\Omega_{B_{l^2(A^{S_1})}} \times \cdots \times \Omega_{B_{l^2(A^{S_n})}}} r_{\mathbf{x}_1} \otimes \cdots \otimes r_{\mathbf{x}_n} d\lambda \\ &= \hat{\lambda}(r_{\mathbf{x}_1} \otimes \cdots \otimes r_{\mathbf{x}_n}), \quad (\mathbf{x}_1, \dots, \mathbf{x}_n) \in B_{l^2(A^{S_1})} \times \cdots \times B_{l^2(A^{S_n})}. \end{aligned} \quad (11.62)$$

This integral representation of  $\eta_{\mathcal{U},\theta}$  is ostensibly weaker than the integral representation of  $\eta_{\mathcal{U},\theta}$  in (11.50).

**Remark 11.8.** A simple  $n$ -linear extension of the Grothendieck inequality ( $n \geq 2$ ), based on  $\theta = 1$  and

$$\mathcal{U} = (\{1\}, \dots, \{n-1\}, \{1, \dots, n-1\}),$$

had appeared in [4], and a subsequent notion of projective boundedness with a prototype of Theorem 10.11, dealing specifically with covering sequences  $\mathcal{U} = (S_1, \dots, S_n)$  of  $[m]$  such that

$$I_{\mathcal{U}} \geq 2 \quad \text{and} \quad |S_i| = k > 0, \quad i \in [n],$$

appeared later in [5].

The dependence of  $\mathcal{K}_{\mathcal{U}}$  on  $k$  and  $n$  was shown in [5] to be  $\mathcal{O}(n^{kn})$ , and was far from optimal. (E.g., compare with (11.58).) To illustrate the significance of  $\mathcal{K}_{\mathcal{U}}$ 's growth, we take

$$\mathcal{U}_n = (\{1, 2\}, \{2, 3\}, \dots, \{1, n\}), \quad n \geq 3, \quad (11.63)$$

and note, via a result in [16], that the algebra of Hilbert-Schmidt operators (the Hilbert space  $l^2(\mathbb{N}^2)$  with matrix multiplication) is a Q-algebra (a quotient of a uniform algebra) if and only if for some  $K > 1$ ,

$$\mathcal{K}_{\mathcal{U}_n} = \mathcal{O}(K^n). \quad (11.64)$$

That the growth of  $\mathcal{K}_{\mathcal{U}_n}$  is indeed given by (11.64) was verified independently in [38] and [23] – in both by an adaptation of the proof of the Grothendieck inequality in [30]. Subsequently in [14], the inequalities involving  $\eta_{\mathcal{U}}$  in [5] were reproved and  $\mathcal{K}_{\mathcal{U}}$ 's growth was shown to be  $\mathcal{O}(K^{kn})$ , via an inductive scheme involving the classical Grothendieck inequality.

## 12. Open questions and loose ends

12.1.  $\tilde{\mathcal{V}}_2(X \times Y)$  **vs.**  $\tilde{V}_2(X \times Y)$ ,  $\mathcal{G}_2(X \times Y)$  **vs.**  $G_2(X \times Y)$ . If  $X$  and  $Y$  are topological spaces, then

$$\begin{aligned}\tilde{V}_2(X \times Y) &\subset C_b(X \times Y) \cap \tilde{\mathcal{V}}_2(X \times Y), \\ G_2(X \times Y) &\subset C_b(X \times Y) \cap \mathcal{G}_2(X \times Y).\end{aligned}\tag{12.1}$$

**Problem 12.1.** *Are the inclusions in (12.1) proper inclusions?*

See §3.1 for definitions, and Remark 3.4 for a brief discussion.

12.2.  $\mathcal{G}_2(X \times Y) = \tilde{\mathcal{V}}_2(X \times Y)$  **vs.**  $G_2(X \times Y) = \tilde{V}_2(X \times Y)$ . The first equality is the Grothendieck theorem for discrete spaces  $X$  and  $Y$ , as per Theorem 2.7, and the second is its "upgraded" version for topological spaces  $X$  and  $Y$ , as per Corollary 3.6.

**Problem 12.2.** *Are the two equalities distinguishable by the respective constants associated with them?*

Namely, we have the universal Grothendieck constant

$$\mathcal{K}_G := \inf \{ \|f\|_{\tilde{\mathcal{V}}_2(X \times Y)} : f \in B_{\mathcal{G}_2(X \times Y)} \},\tag{12.2}$$

where  $X$  and  $Y$  are infinite sets with no *a priori* structures, and otherwise, if  $X$  and  $Y$  are topological spaces, then we have

$$\mathcal{K}_{GC}(X \times Y) := \inf \{ \|f\|_{\tilde{\mathcal{V}}_2(X \times Y)} : f \in B_{G_2(X \times Y)} \}.\tag{12.3}$$

For infinite  $X$  and  $Y$ ,

$$\mathcal{K}_G \leq \mathcal{K}_{GC}(X \times Y) \leq K^2,\tag{12.4}$$

where  $K$  is the constant in (3.11), and the problem becomes: is  $\mathcal{K}_G < \mathcal{K}_{GC}(X \times Y)$ ?

12.3. **(weak  $l^2 \rightarrow$  weak  $L^2$ )-continuity.** The map  $\Phi$  (of Theorem 3.5) is not continuous with respect to the weak topologies on  $l^2(A)$  and  $L^2(\Omega_A, \mathbb{P}_A)$  (and *a fortiori* with respect to the weak\*-topology on  $L^\infty(\Omega_A, \mathbb{P}_A)$ ). For, if  $\mathbf{x}_j \rightarrow \mathbf{x}$  weakly in  $l^2(A)$ ,  $\mathbf{x} \neq \mathbf{0}$ , and

$$\liminf_j \|\mathbf{x}_j\|_2 > \|\mathbf{x}\|_2,$$

then

$$\widehat{Q_A(\mathbf{x}_j)} \rightarrow \widehat{Q_A(\mathbf{x})} \text{ on } W_A \setminus R_A,$$

and therefore

$$\|\mathbf{x}_j\|_2 \widehat{Q_A(\mathbf{x}_j)} \rightarrow \|\mathbf{x}\|_2 \widehat{Q_A(\mathbf{x})} \quad \text{on } W_A \setminus R_A.$$

**Problem 12.3.** *Is there a map from  $l^2(A)$  into  $L^2(\Omega_A, \mathbb{P}_A)$ , which is continuous with respect to the weak topologies on its domain and range, and also satisfies (3.11) and (3.13) (in Theorem 3.5)?*

The problem arises in connection with integral representations of scalar-valued functions on  $X \times Y$  that are continuous separately in each coordinate. (See the discussion leading to Definition 3.1.) Let  $\mathbf{g} = \{g_\omega\}_{\omega \in \Omega}$  be a family of scalar-valued functions defined on a topological space  $X$  and indexed by a finite measure space  $(\Omega, \mu)$ . We say that  $\mathbf{g}$  is *weakly- $L^2(\mu)$ -continuous*, if for each  $x \in X$

$$\omega \mapsto g_\omega(x), \quad \omega \in \Omega,$$

determines an element of  $L^2(\Omega, \mu)$ , and the resulting map  $\mathbf{g} : X \rightarrow L^2(\Omega, \mu)$  defined by

$$\mathbf{g}(x)(\omega) = g_\omega(x), \quad \omega \in \Omega, \quad x \in X,$$

is continuous with respect to the weak topology on  $L^2(\Omega, \mu)$ . Indeed, if in Questions 2.3 and 2.6, the underlying domains  $X$  and  $Y$  are topological spaces and the representing families are weakly- $L^2$ -continuous, then the subsequent integral representations in (2.13) and (2.22) determine functions that are continuous separately in each coordinate. Specifically, if a Hilbert space  $H$  is viewed through the weak topology, then its inner product  $\langle \cdot, \cdot \rangle_H$  is continuous in each coordinate separately, but not jointly, and hence the question: does  $\langle \cdot, \cdot \rangle_H$  have an integral representation wherein integrands are uniformly bounded in the  $L^\infty$ -norm and also weakly- $L^2$ -continuous? In this connection, we obtain from §6.4 that if  $L^2(\Omega_A, \mathbb{P}_A)$  in Problem 12.3 is replaced by a quotient of  $L^2(\Omega_A, \mathbb{P}_A)$ , then the answer to a modified, less stringent question is affirmative.

**12.4. Projective boundedness vs. projective continuity.** Every projectively continuous functional is projectively bounded, and in all known instances (supplied by Theorem 10.11), every projectively bounded functional is also projectively continuous.

**Problem 12.4** (cf. Problem 12.1). *Is every projectively bounded multilinear functional on a Hilbert space also projectively continuous?*

**Problem 12.5** (cf. (10.16), Theorem 10.11, Problem 12.2). *For integer  $m > 0$ , let  $\mathcal{U} = \{S_1, \dots, S_n\}$  be a covering sequence of  $[m]$  with  $I_{\mathcal{U}} \geq 2$ . Let  $A$  be an infinite set, and let  $\eta_{\mathcal{U}}$  be the  $n$ -linear functional on  $l^2(A^{S_1}) \times \dots \times l^2(A^{S_n})$  defined in (11.1). Let  $\mathbf{B}^{[\mathcal{U}]} := B_{l^2(A^{S_1})} \times \dots \times B_{l^2(A^{S_n})}$ . Is*

$$\|\eta_{\mathcal{U}}\|_{\tilde{V}_n(\mathbf{B}^{[\mathcal{U}]})} < \|\eta_{\mathcal{U}}\|_{\tilde{V}_n(\mathbf{B}^{[\mathcal{U}]})}^? \quad (12.5)$$



**12.5. A characterization of projective boundedness.** Let  $\eta$  be a bounded  $n$ -linear functional on a Hilbert space  $H$  with an orthonormal basis  $A$ , and let  $\theta_{A,\eta}$  be its kernel relative to  $A$ , as defined in (1.36). If  $\eta$  is projectively bounded, then  $\theta_{A,\eta} \in B((R_A)^n)$ , i.e., there exists a complex measure  $\lambda \in M((\Omega_A)^n)$  such that

$$\eta(\alpha_1, \dots, \alpha_n) = \hat{\lambda}(r_{\alpha_1} \otimes \dots \otimes r_{\alpha_n}), \quad (\alpha_1, \dots, \alpha_n) \in A^n. \quad (12.6)$$

(See Remark 10.8.ii.)

**Problem 12.6.** Suppose  $\theta_{A,\eta} \in B((R_A)^n)$ . Is  $\eta$  projectively bounded?

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